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Elementary Functions and Complex Numbers

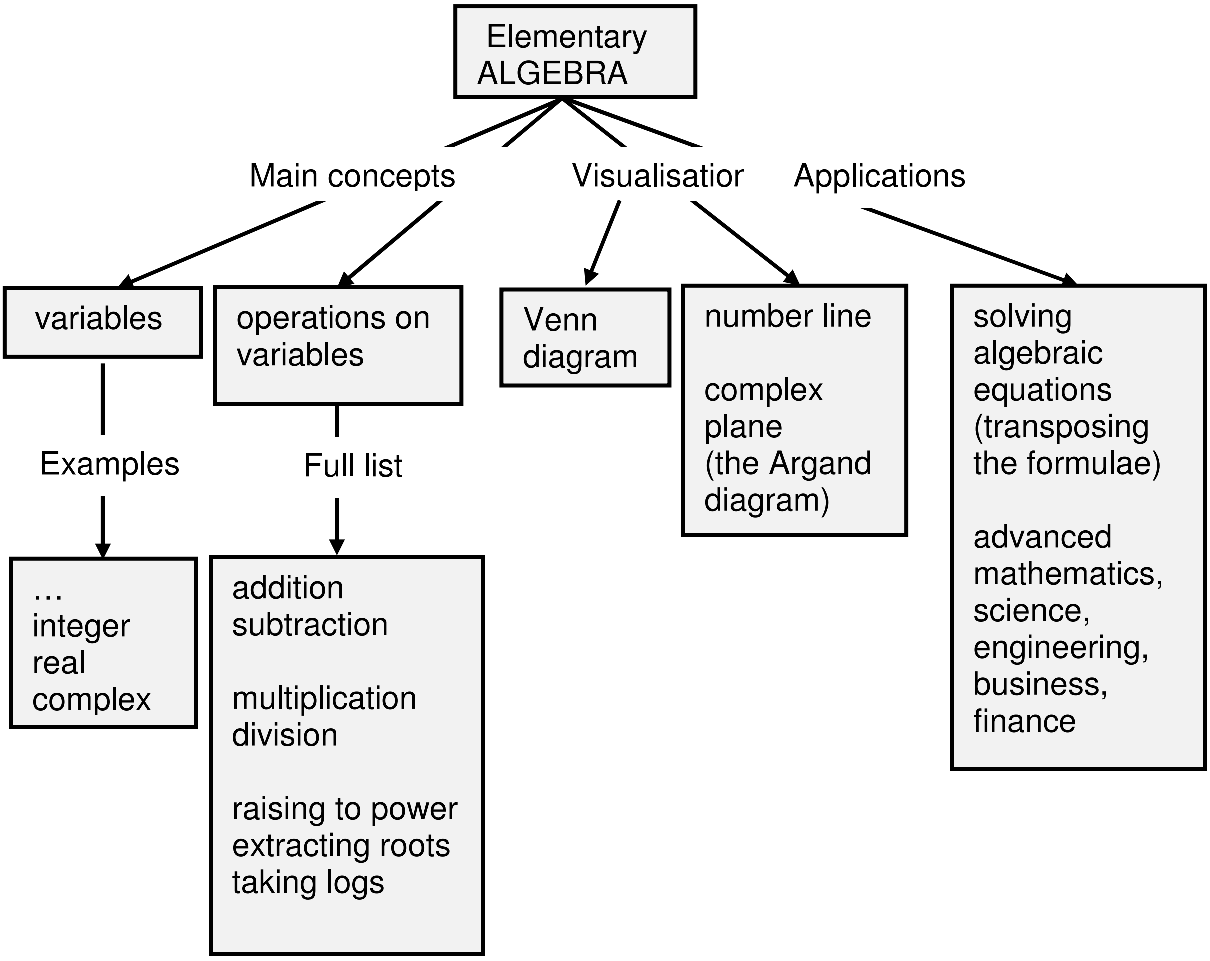
I. Table of Contents

II. CONCEPT MAPS

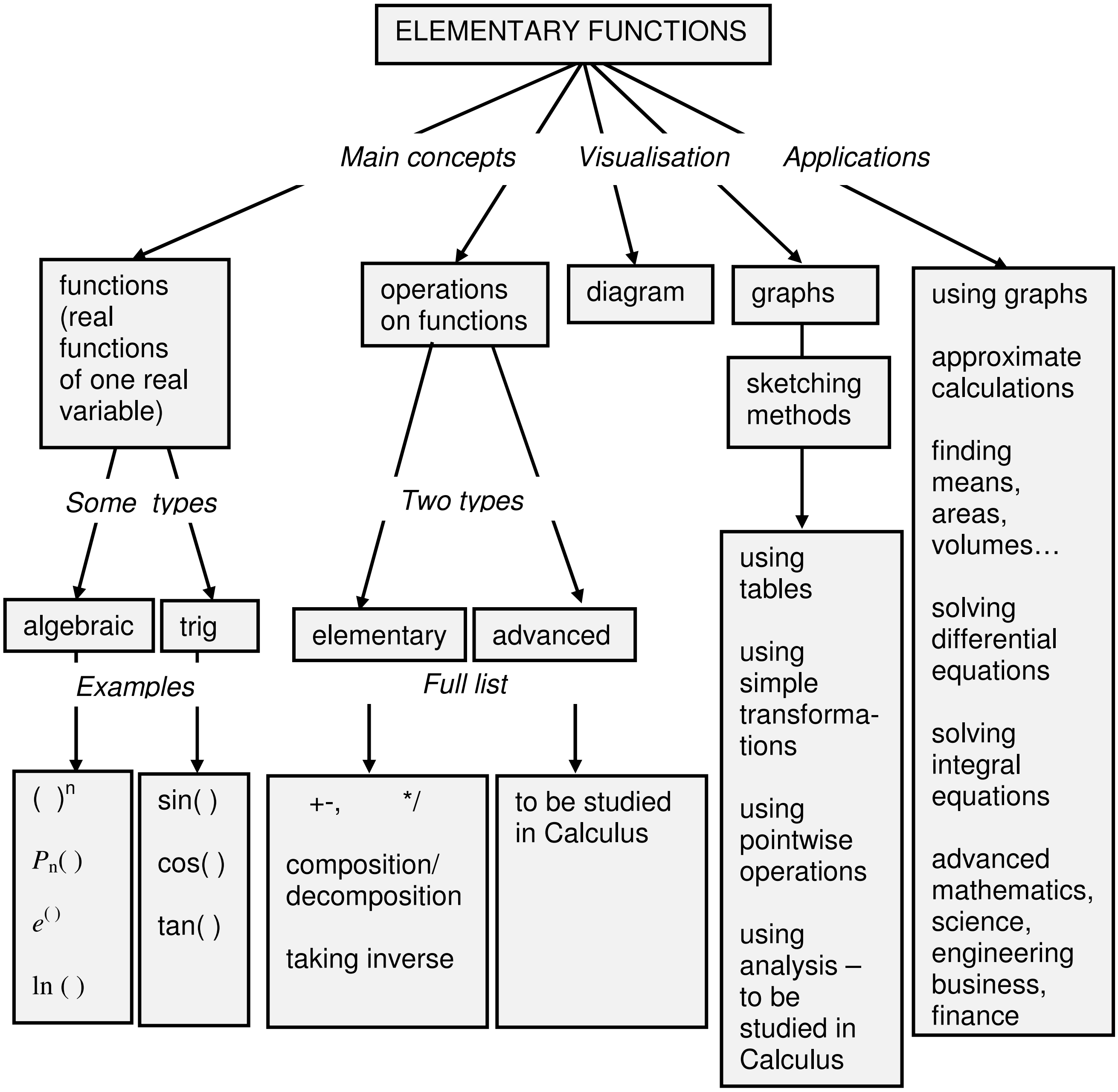
Throughout when we first introduce a new concept (a technical word or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea.

Two major topics are covered in these Notes, Elementary Functions and Complex Numbers. You can understand these topics best of you first study the Notes on Elementary Algebra.

Here is a concept map of Elementary Algebra.



Here is a concept map of of the Functions part of Pre-Calculus. It is best to study it before studying any of the Lectures to understand where they are on the map. The better you see the big picture the easier it is to absorb this material!



II. LECTURES

Lecture 4. FUNCTIONS

We generalise the concept of an operation on variables as introduced in Algebra by introducing functions.

4.1 Variables

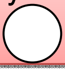
As stated in Section 1.1,

A **variable** is an **abstraction** of a quantity.
In calculus, variables are denoted mostly by $x, y, z, p, \ldots.. w$.

Abstraction is a general concept formed by extracting common features from specific examples. Specific examples of a quantity are time, distance, magnitude of force, current, speed, concentration, profit and so on.

A variable can be dimensional, that is, measured in dimensional units $s, m, N, A, m/s, kg/m^3$, £… or non-dimensional units, that is, units taking any **value** from an allowed **set of numbers**.

Diagrammatically any set can be represented (visualised) as a circle (this circle is called a **Venn diagram**). It might help you to think of this circle as a bag containing all elements of the set.



Examples of commonly used sets of numbers in science and engineering:

I – set of all integers
R – set of all real numbers

All numbers that can be represented **graphically** by points on a number line are called **reals**.

A **real variable** x is a variable that takes only real values, **maybe with some exceptions** (that is, for some reason some real values are excluded). Describing the set of all such values we can say "x real" or "$x \in R$"

this symbol means **"element of"**

4.2 Functions

A **function** represents a relationship between variables. In elementary calculus we study functional relationships between two variables only, one of them is called an **independent variable**, **input** or a **control variable** and the other, **dependent variable** or **output**. The independent variable is also called an **argument** and dependent variable is also called a **function of** independent variable.

The first choice for a function symbol is f or $f\,(\ )$, the second choice is g or $g\,(\ )$, and the third choice is h or $h\,(\ )$. The brackets put after the function symbol stand for the word **of** and are **functional** and **not algebraic**, which means that **no multiplication sign is implied!** Note that the word *function* has two meanings:

1. $f(\ )$ **is an operation or a chain of operations** on an independent variable;
2. $f(x)$ is a **dependent variable**, that is the variable obtained when an operation or a chain of operations $f(\ )$ is applied to an independent variable x.

Which meaning is implied should be understood from the context; if the word *function* is not followed by the word *of* it is most likely to be an operation, otherwise it is most likely to refer to a dependent variable.

Examples:

1. $f\,(x) = x + 2$

   This line is verbalised as follows: f **of** x is x plus 2. The operation $f\,(\ )$ is "+2".

   $\Rightarrow\ f\,(1) = 3$

   This line is verbalised as follows: f **at** 1 is 3.

2. A **sequence** $\{x_n\} = x_1, x_2, x_3, \ldots$ is a function of integer n, that is, we can write

$$x_n = x(n)$$

   A simple example of a sequence is $x_n = n\ (x_1 = 1, x_2 = 2, x_3 = 3, \ldots)$.

3. $f\,(x) = |x|$, where the **absolute value (modulus)** of x is

$$|x| \;=\; \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

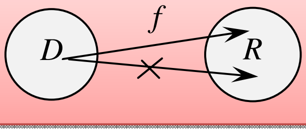
   This definition should be read line by line from top to bottom, but – unusually - each line should be read from right to left: If x is positive or zero, then its absolute value is itself. If x is negative, then its absolute value is its additive inverse, e.g. $|2| = 2$ and $|-2| = 2$.

   Note that the **geometrical interpretation** (visualisation) of the modulus based on the number line is the **distance** (from the point representing the number on this line) **to the origin** (the point representing 0).

   Question: What is modulus of $-\frac{1}{3}$?

To specify a function we need to specify a (chain of) operation(s) and **domain** D (a set of allowed values of the independent variable). To each $x\in D, f(\,)$ assigns **one and only one** value $y\in R$, where R is **range** (a set of all possible values of dependent variable).

A diagrammatical representation of a function



In Elementary Calculus we study only real functions of a real variable (that is, of one real argument).

A **real function** has the range that is a set of real numbers. The **natural domain** of any **function of a real variable** is reals, **with possible exceptions. Exceptions arise due to operations that might not lead to a real number**, such as division by zero, taking the square root (in fact, any even root) or a log. So, a real function of a real variable is a function whose range and domain are both real.

4.3 Elementary operations on functions

1. ADDITION (INCLUDING SUBTRACTION)

$(f \pm g)(x) = f(x) \pm g(x)$ if $D_f = D_g$ (that is, if the functions have the same domain)

2. MULTIPLICATION

$(f \cdot g)(x) = f(x) \cdot g(x)$ if $D_f = D_g$

3. DIVISION

$\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ if $D_f = D_g$, $g(x) \neq 0$

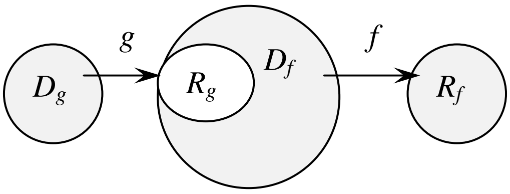
4. COMPOSITION

$f\circ g(x) \equiv f(g(x))$ if $R_g \subseteq D_f$

symbol of composition

"is by definition"

"is subset of"



Composition is the chain of functions (operations) applied one by one and amounts to substitution: when performing composition we keep putting in place of arguments their actual expressions. This means that we keep removing the brackets inside out.

Example: $f(x) = x + 2,\ \ g(x) = x^2$.

1. Find the composition $f(g(\mathrm{x}))$.

Solution

Step 1.

$f(g(x)) = f(x^2)$ (in place of $g(x)$ we put x^2)

Step 2.

$f(x^2) = x^2 + 2$ (operation f is $+2$, that is *add* 2 *to whatever you have in brackets*.)

2. Find the composition $g(f(x))$.

Solution

Step 1.

$g(f(x)) = g(x+2)$ (in place of $f(x)$ we put $x+2$)

Step 2.

$g(x+2) = (x + 2)^2$ (operation g is $(\ \ )^2$, that is, *square whatever you have in brackets*.)

We often have to perform an operation of **DECOMPOSITION** (inverse to composition) by sorting out the order of operations in a given expression.

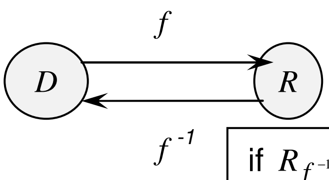
Examples:

1. Decompose the function $h(x) = x^2 + 2$.

Solution
Using the Order of Operations,
1st operation is $(\ )^2$ (power 2)
2nd operation is $+2$.

2. Decompose the function $p(\mathrm{x}) = (x + 2)^2$

Solution
Using the Order of Operations,
1st operation is $+2$
2nd operation is $(\ )^2$ (power 2)



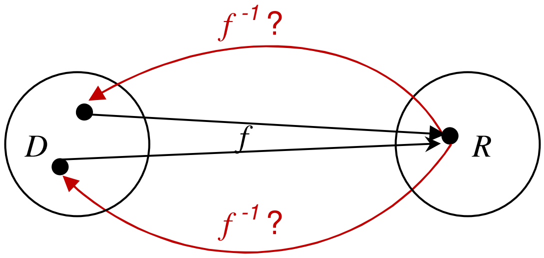
if $R_{f^{-1}} \subseteq D_f$
if $R_f \subseteq D_{f^{-1}}$

5. TAKING INVERSE OF A FUNCTION

$f^{-1}(x) : f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$

symbol of inverse function.
Not to be confused with the reciprocal!

Note: the inverse function does not always exist (see the diagram below). HERE



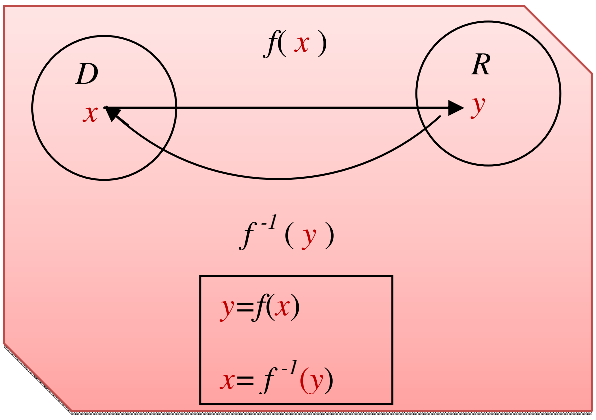
Finding an inverse function (when it exists)

To find an inverse functions use the following **algorithm** (a sequence of steps):

Step 1. rename $f(x)$ as y

Step 2. solve for x

Step 3. rename x as $f^{-1}(y)$



Example: $f(x) = x + 2$. Find $f^{-1}(x)$.

Solution

Step 1. Rename $f(x)$ as y
$\Leftrightarrow y = x + 2$
Step 2. Solve for x
$\Leftrightarrow x = y - 2$
Step 3. Rename x as $\mathrm{f}^{-1}(\mathrm{y})$
$\Leftrightarrow f^{-1}(y) = y - 2$
Step 4. Rename y as x
$\Leftrightarrow f^{-1}(x) = x - 2$

4.4 Order of Operations (extended)

B	f ( )	P	M	A
including		powers	×	+
implicit		roots	÷	-
		logs		

Note: all operations related to Powers can be considered to be advanced algebraic operations as well as being functions (see below).

4.5 Applications of real functions of real variable and operations on such functions

Any analogue signal representing a physical quantity, such as voltage, current, speed, force, temperature, concentration *etc.* can be considered to be a real function of time (which is a real variable). Many instruments, such as controllers transform signals, that is, perform operations on functions.

Example: Flow-rate (Q) is the rate at which the water, in terms of volume, flows along a channel (see figure 4.1). It is expressed as volume divided by time, in units of m^3/s (cubic metres per second). If we assume that flow in a channel in uniform, then the flow-rate is a function of:

- the area of flow in the channel (m^2)
- the hydraulic radius (m)
- the slope of the channel (in the direction of flow)
- the roughness the channel material that is in contact with the water

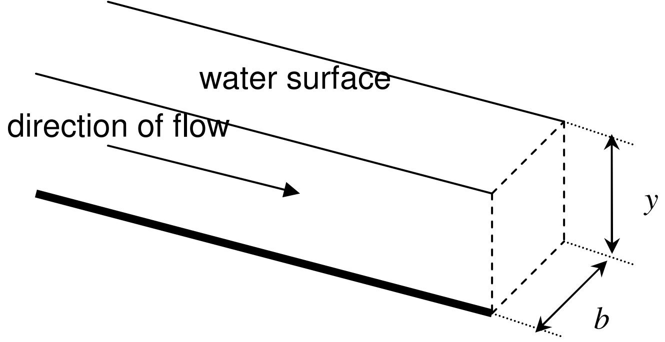


Figure 4.1. The longitudinal section of a channel

Specifically this can be represented in the "Manning equation":

$$Q = \frac{A}{n} R^{2/3} S^{1/2}$$

where A = cross-sectional area

R = hydraulic radius; the cross-sectional area divided by wetted perimeter
$= A / P$, with P = 'wetted perimeter': the length of contact between the water and the channel $(y + b + y)$ for the rectangle)
S = channel slope in the direction of flow
n = Manning coefficient of roughness

You have been employed on the design of the irrigation scheme intended to provide water to a number of rural villages where clean water for drinking and for agricultural purposes is lacking. It is proposed to provide open 0.9m wide rectangular-section channels lined with concrete with the water flowing to a maximum depth of $1.2\ m$. It is required to have a capacity of $2.0\ m^3/s$ (the highest flow-rate it is expected to carry). If the channel is constructed using rough finished concrete ($n = 0.017$), what slope (in the direction of flow) will be needed assuming that the flow is uniform? If the fall of the land is fairly shallow and the maximum fall of the channels for economic construction is restricted to 0.3m per 100 metres is this design adequate?

http://www.raeng.org.uk/education/diploma/maths/pdf/CBE7_Channel_Flow.pdf

4.6 A historical note

The idea of a function dates back to the Persian mathematician, Sharaf al-Dīn al-Tūsī, in the 12th century" who used this word in the sense of the dependent variable as did Gottfried Leibniz in end of 17th and beginning of the 18th century and Leonhard Euler in the 18th century. It seems that a more formal definition of a function as a relation in which every element from a domain has a unique element from the range has been given in the 19th century by Lobachevsky. This is close to the above definition of a function as an operation on an independent variable.

4.7 Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and ORDER OF OPERATIONS**
- **Revise ALGEBRA Lecture 2 and study Solutions to Exercises in Lecture 2 using the STUDY SKILLS Appendix**
- **Revise ALGEBRA Lecture 3 using the STUDY SKILLS Appendix**
- **Study Lecture 4 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Let $f(t)=\frac{9}{5}t+32$. This function converts temperature from ^{0}C to ^{0}F. Evaluate

a) $f(0)$
b) $f(100)$
c) $f(24)$

Q2. The speed $v(t)$ of an object is given by $v(t)=98t$. Find

a) $v(t^2)$
b) $v(t+1)$

Q3. Decompose the following functions (that is, state what is the first operation in $f(x)$ and what is the second. Use the Order of Operations Summary).

a) $f(x)=\frac{1}{x+2}$
b) $f(x)=\frac{1}{x}+2$
c) $f(x)=\frac{1}{2}(x+2)$
d) State which of these can be called a sum, a quotient or a product.
e) What are the factors in this product?

Q4. Given function $f(x)=\frac{1}{x+2}$

a) state what is the natural domain of $f(x)$
b) find the inverse function $f^{-1}(x)$
c) what is its natural domain?

Additional Exercises

1. Turn into one power
   a) $2^3 \times 2^5$
   b) $c^3 \times c^5$
   c) $2^{2+j} \times 2^{2-j}$
   d) $\dfrac{s^{1-\sqrt{2}}}{s^{2+\sqrt{2}}}$
   e) $\dfrac{e^{1-j\sqrt{2}}}{e^{2+j\sqrt{2}}}$

2. Modify into a power with positive index
   a) a^{-3}
   b) u^{-1}
   c) w^{-10}

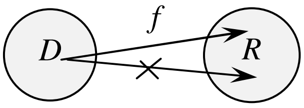
3. Evaluate and check your result:
   a) $\sqrt{4}$
   b) $\sqrt{9}$
   c) $\sqrt{16}$
   d) $\sqrt{25}$
   e) $36^{1/2}$
   f) $49^{1/2}$
   g) $8^{1/3}$
   h) $27^{1/3}$
   i) $\sqrt[3]{64}$
   j) $\sqrt[3]{125}$

4. Evaluate without a calculator:
   a) $\log_{10} 2 + \log_{10} 5$
   b) $\log_{10} 200 - \log_{10} 2$
   c) $\log_{10} 10000$
   d) $\log_2 8$
   e) $\log_2 \dfrac{1}{2}$
   f) $\log_{10} \dfrac{1}{100}$

Lecture 5. Real FUNCTIONS of One Real Variable: Graphs, Polynomials

5.1 Graphical representation of real functions of one real variable

In the last lecture we gave a diagrammatical representation of a function,



Each particular real function of a real variable can also be visualised using a more specific, **graphical**, representation. On a **graph** of a function $y = f(x)$,

domain D is represented by a set of points on a **horizontal axis** (number line), maybe not the whole axis, maybe an interval; it is usually called the x-axis;

range R is represented by a set of points on a **vertical axis** on a plane, maybe not the whole axis, maybe an interval; it is usually called the y-axis;

function f (in the sense of *operation*) is represented by collection of points, maybe a line. Each point on this line has **coordinates** x and y (projections onto x- and y-axis, respectively) which satisfy the **equation of the line**. This means that the x-coordinate *of* **each point on this line** is related to the y-coordinate of this point via the functional relationship $y = f(x)$.

Graphs can be **sketched by using the table** of pairs of related values of independent and dependent variable. Each pair describes a point on the graph.

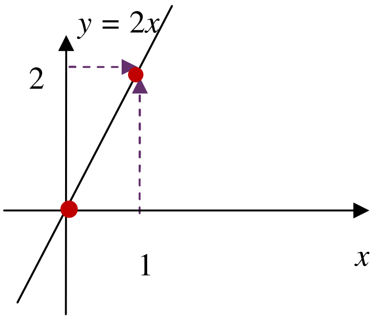
Examples:

1. $y = 2x$

Table:

x	0	1	…
y	0	2	…

Graph:

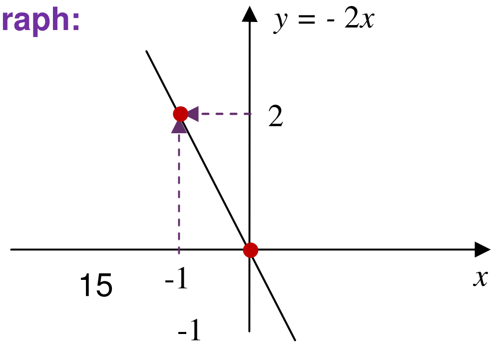


2. $y = - 2x$

Table:

x	0	-1	…
y	0	2	…

Graph:

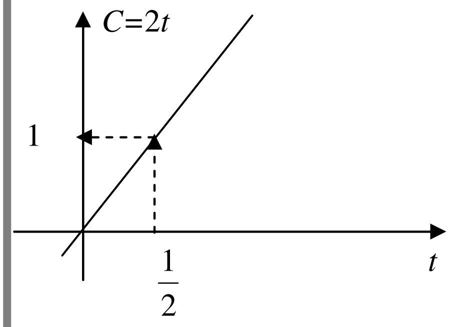


5.2 How to use graphs of real functions $y = f(x)$ of one real variable

The graph may be used as a rule (an operation or a chain of operations), which, given an x, allows one to find the value of y and *vice versa*.

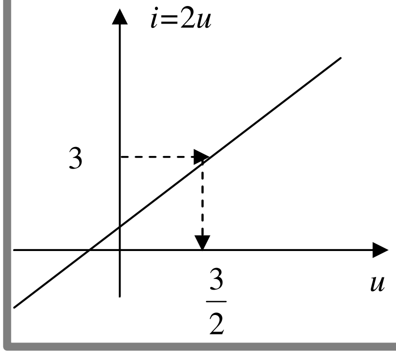
Examples:

1. $C = 2t.$ $C(\frac{1}{2})$? (**verbalise:** given $t = \frac{1}{2}$ find C)



Answer:

2. $i = 2u,$ $u = i^{-1}(3)$? (**verbalise:** given $i = 3$ find u)



Answer:

5.3 Applications of graphs

Specifications of various instruments often contain graphs showing say, the device output at various frequencies (see figure 5.1). Properties of various devices and materials can also often be summarised as graphs (see figure 5.2).

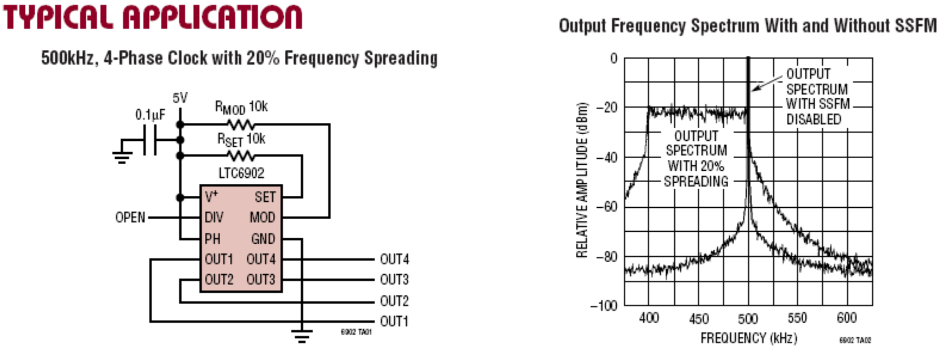


Figure 5.1. Copied from description of an oscillator at http://www.linear.com/pc/downloadDocument.do?navId=H0,C1,C1010,C1096,P2293,D1568

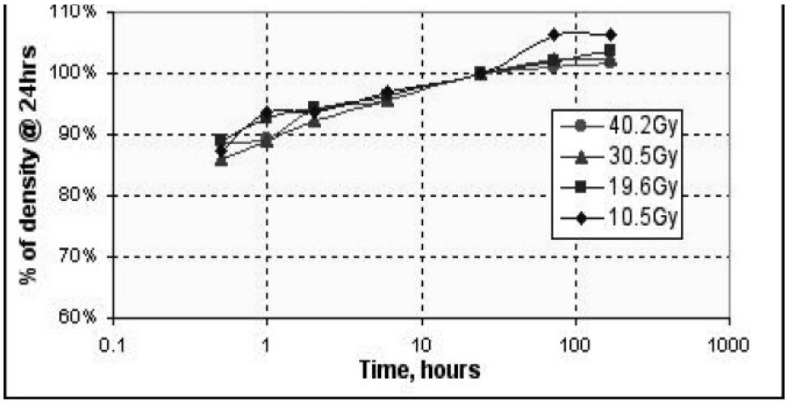


Figure 5.2. Copied from description of a dosimetry film at **http://www.ispcorp.com/products/dosimetry/content/gafchromic/content/products/md55/pdfs/conspefo.pdf**

5.4 Elementary functions: monomials (natural powers) and polynomials

Elementary functions involve only elementary (algebraic and trigonometric) operations.

A **polynomial** is an elementary function, involving only algebraic operations of addition and multiplication by a constant of **monomials** (natural power functions),

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots.+a_0,$$

where n is a natural number and coefficients (factors) $a_0, a_1, \ldots., a_n$ are constants (independent of x). **Verbalise:** A polynomial is a sum. Each term in this sum is a product of a whole power of x and a constant. If $a_n \neq 0$, n is called the **degree** of $P_n(x)$.

Question: What is a constant?
Answer:

To be precise, we say that $a_0, a_1, \ldots., a_n$ are **constant with respect to** x.

5.4.1 The Main Theorem of Algebra

The Main Theorem of Algebra (without proof)

The polynomial equation $P\mathrm{n}(x) = 0$ has exactly n roots (solutions), $\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}$, which may be equal or complex (not real).

5.4.2 Factorisation of polynomials

Each polynomial may be factorised

$$P_n(x) = a_n(x - x_1)(x - x_2)...(x - x_n)$$

Note: the derivation of the above formula lies outside the scope of these notes but you can check yourself that it is correct at the roots by substituting x_1 for x, x_2 for x *etc.* You can also check that both LHS (left-hand side) and RHS (right-hand side) of this formula have the same coefficient a_n associated the highest power x^n.

In most cases roots (solutions) of polynomial equations can be found only by using a computer code. However, as discussed in Section 3.2.1, for any quadratic equation $ax^2 + bx + c = 0$ we have a formula for finding roots:

$$ax^2 + bx + c = 0$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For cubic equations, roots can be found without the help of a computer only in isolated cases. For example, $x^3 - 3x^2 + 3x - 1 = (x - 1)^3$. Therefore, the equation

$$x^3 - 3x^2 + 3x - 1 = 0$$

has three equal roots, $x_1 = x_2 = x_3 = 1$.

5.4.3 Graphs of Polynomials of Degree 1 (linear), 2 (quadratic) and 3 (cubic)

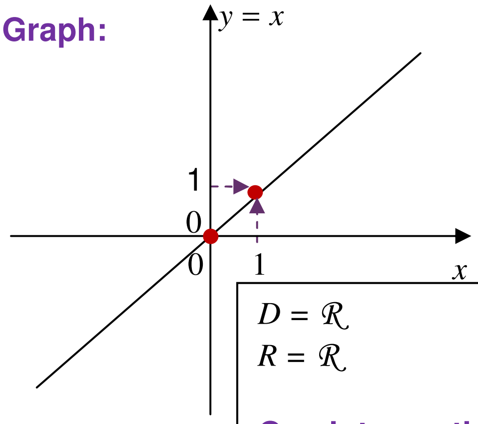
1. A polynomial of degree 1, $P_1(x)$.is called a **linear function**. Its graph is called a **straight line.**

Example: Sketch a **monomial** (power function) $y = x$

Table:

x	0	1
y	0	1

Graph:



$D = \mathcal{R}$
$R = \mathcal{R}$

One intersection with the x-axis

NOTE: to draw a straight line all you need is two points. More points are required to sketch other graphs correctly.

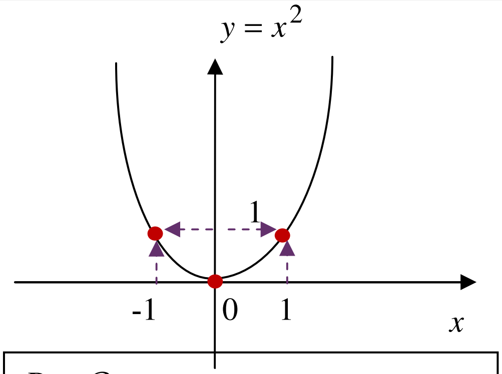
2. A polynomial of degree 2, $P_2(x)$ is called a **quadratic function**. Its graph is called a **parabola**.

Example: Sketch a monomial $y = x^2$

Table:

x	0	1	-1	2	-2
y	0	1	1	4	4

Graph:



$D = \mathcal{R}$

R: $y \geq 0$

One intersection with the x-axis (two roots coincide)

Note: any parabola is symmetric with respect to the vertical line crossing it at its tip (which is a turning point).

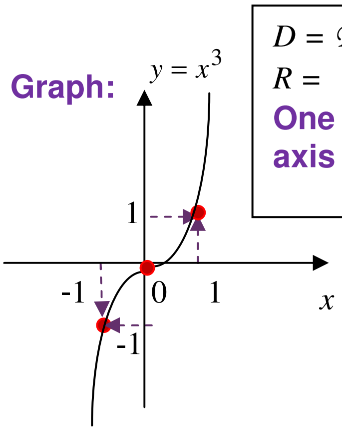
3. A polynomial of degree 3, $P_3(x)$.is called a **cubic function**. Its graph is called a **cubic parabola**.

Example: 1. Sketch a monomial $y = x^3$

Table:

x	0	1	-1	2	-2
y	0	1	-1	8	8

Graph:



$D = \mathcal{R}$

$R = \mathcal{R}$

One intersection with x-axis (three roots coincide)

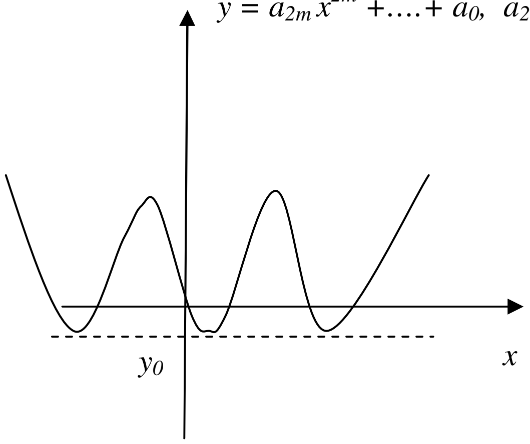
Note: At each intersection with the x-axis, y=0. Therefore, the intersections represent roots (solutions) of the equation y=0.

Optional

5.4.4 Graphs of General Polynomials of Even and Odd Degree

The general polynomial of even degree $2m$, where m is a natural number

$y = a_{2m}x^{2m} + \dots + a_0,\ \ a_{2m} > 0$



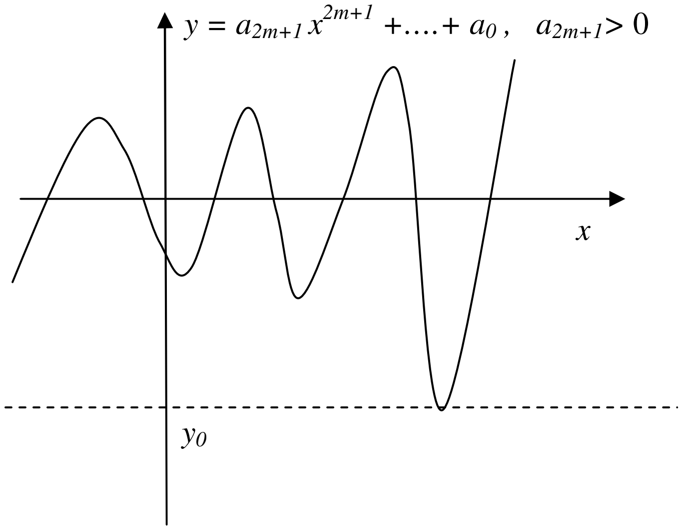
$D = \mathcal{R}$
R: $y \geq y_0$

Both branches up

No more than $2m$ intersections with the x-axis

The general polynomial of odd degree $2m+1$, where m is a natural number

$y = a_{2m+1}x^{2m+1} + \dots + a_0\ ,\ \ a_{2m+1} > 0$



$D = \mathcal{R}$
R: $y \geq y_0$
Left branch down, right branch – up

No more than $2m+1$ intersections with the x-axis

5.5 Instructions for self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise Lectures 3 and study Solutions to Exercises in Lecture 3 using the STUDY SKILLS Appendix**
- **Revise Lecture 4 using the STUDY SKILLS Appendix**
- **Study Lecture 5 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. If $F(s) = \dfrac{1}{s+1}, s \neq -1$, find

a) $F(s-1)$

b) $F(s+1)$

c) $F(\frac{1}{s^2+\omega^2})$

Q2. If $f(t)=\frac{1}{t-3}, t \neq 3$ and $g(t)=\frac{1}{t+3}, t \neq -3$ what are the following functions (do not forget to describe their natural domains):

a) $f(g(x))$

b) $g(f(x))$

c) $f(f(x))$

Q3. Find the inverse of the function $g(z)=\frac{z-1}{z+1}, z \neq -1$. Indicate its natural domain.

Q4.

a) Plot a graph of the function $T_F=\frac{9}{5}T_C+32,\ 0 \le T_C \le 100$. What is the range of this function?

b) Sketch the function $f(x)=(x-2)(x+4)$. What is the natural domain of this function? What is the natural range of this function?

Additional Exercises

1. $f(x) = x^2 - 3x + 2$ (no domain indicated means that domain is R, the set of all reals). Find $g(t)=f(\sqrt{t})$ and indicate the natural domain of $g(t)$.
2. Find the natural domain of $f(x)=\frac{1}{\mathrm{x}}$.
3. Find the natural domain of
   a) $\sqrt{\mathrm{x}-2}$
   b) $\sqrt{t^2-3t-2}$
   c) $\sqrt{-t^2+3t+2}$
4. Sketch $y = 2(x-1)(x-2)(x-3)$ using only information from Lectures 1 - 5.
5. Sketch $y = \frac{1}{2}x^2+\frac{1}{4}x+5$ using only information from Lectures 1 - 5.

Lecture 6. Real FUNCTIONS of One Real Variable: Exponential Functions, Logarithmic Functions, Inverse Functions

6.1 Exponential functions

In algebra we studied the operation of *raising to power*. It can be used to introduce **exponential functions** $f(x) = b^x$, $b > 0$ (compare to monomial functions $f(x) = x^n$ with n being a whole number, which were covered in Lecture 5).

Note: the position of symbols – that is, whether they are on the main line, lowered or raised – is very important in mathematics.

Often it is convenient to choose the positive number b to be $e \approx 2.71$ (**Neper's constant**, an irrational number). Why is it convenient is discussed later.

Question: What is a rational number?
Answer:

Question: What is an irrational number?
Answer:

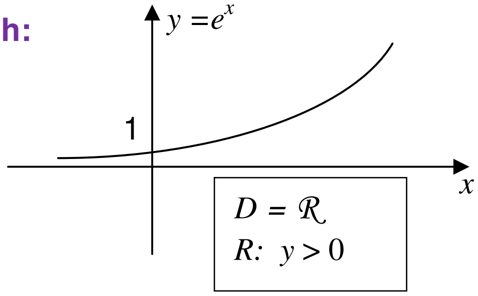
Question: Can we be more specific?
Answer:

Let us use the table to sketch the graph of the exponential function $y = e^x$:

Table:

x	0	1	-1	2	-2
y	1	$e \approx 3$	$\frac{1}{e} \approx \frac{1}{3}$	$e^2 \approx 9$	$\frac{1}{e^2} \approx \frac{1}{9}$

Graph:



6.2 Logarithmic functions

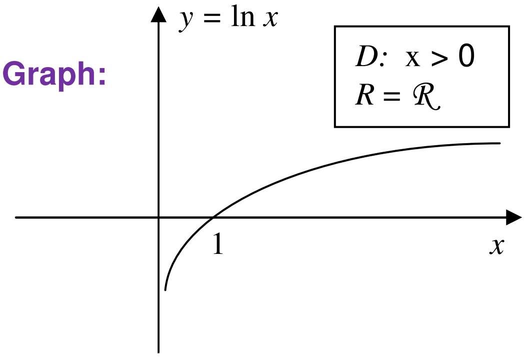
In algebra we learned that the inverse operation to *raising b to power* is called $\log_b$. It is defined as follows: $\log_b x = p$: $b^p = x$, so that $\log_b x$ is the power (index, exponent) to which you have to raise b to get x. We now introduce a **logarithmic function** $f(x) = \log_b x$. Note that the function (operation) $\log_b(\ )$ is inverse of the exponential function (operation) $b^{(\ )}$. This means that when executed in sequence these two operations cancel each other.

If $b = e$, the logarithmic function is usually written as $f(x) = \ln x$ (this is verbalised as the **natural log of** x). Note that the function $\ln(\ )$ is inverse to the function $e^{(\ )}$. We could use a table to sketch the log function and obtain

Table:

x	1	e	e^{-1}	e^{2}	e^{-2}
y	0	1	-1	2	-2

Graph:



D: x > 0
$R = \mathcal{R}$

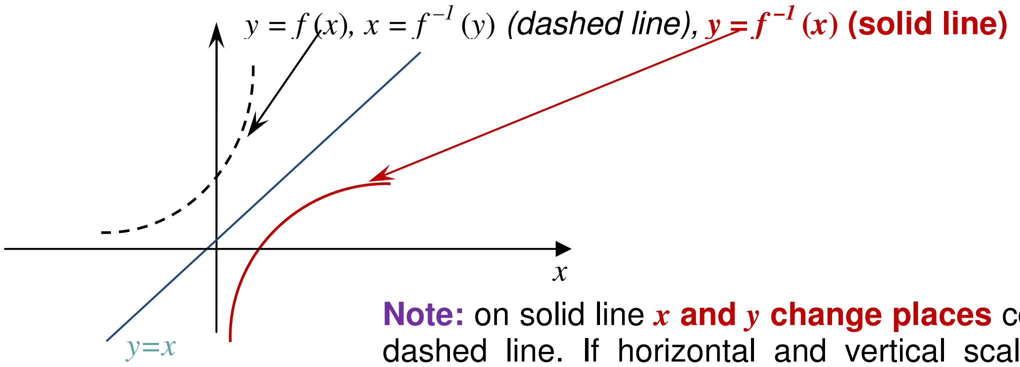
A much better way of sketching the log functions is to use the following general algorithm for sketching inverse functions:

SKETCHING AN INVERSE FUNCTION $y = f^{-1}(x)$

Step 1. Sketch the graph of function $y = f(x)$.
Note that it is also the graph of inverse function $x = f^{1}(y)$.

Step 2. Sketch the "mirror" line $y = x$.
Note that every point on this line has y*-coordinate the same as* x*-coordinate.*

Step 3. If horizontal and vertical scales are the same, sketch the mirror image $y = f^{1}(x)$ of the original graph $y = f(x)$ with respect to the line $y = x$.



Note: on solid line x **and** y **change places** compared to dashed line. If horizontal and vertical scales are the same this corresponds to reflection of the dashed line with respect to the line $y = x$.

6.3 Revision: Trigonometry

6.3.1 Angles, Degrees and Radians

Let us consider a segment of length r and rotate it around a fixed end until it returns to the original position. The angle described after a full rotation is said to be 360° (**Verbalise:** 360 **degrees**). The degree is a **dimensional unit of angle**.

Note: the degree is a dimensional unit in the sense that it has a convenient but arbitrarily chosen size (dimension). For example, we could just as well subdivide the full rotation into 720 units and call the new unit of angle *a degree*.

The line described by the other end of the segment is called **circumference**. If this end does not complete the full rotation the line is called a **circular arc** s – see figure 6.1 below.

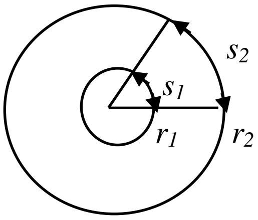


Figure 6.1. Circumferences of radii r_1 and r_2 and circular arcs s_1 and s_2.

It turns out that whatever the length of the segment (which is called the **radius** of this circumference), the quotient

$$\frac{\text{length of circumference}}{\text{radius}}$$

is always the same number called 2π, where π (a Greek letter *pi*) is a shorthand for an irrational number $\pi \approx 3.14$. By the same token, whatever the radius the quotient

$$\frac{\text{length of circular arc}}{\text{radius}}$$

is always the same number. This number can be taken to be a dimensionless measure of the corresponding angle, without involving any units. Yet, for historical reasons, it is customary to call this number **radian**, the dimensionless unit of angle. So, while we *write* that the angle of full rotation is 2π, we sometimes *say* that the angle of full rotation is 2π radians:

$$\Rightarrow 2\pi \text{ (rad)} = 360° \quad / \div 2\pi$$

$$\Rightarrow 1 \text{ (rad)} = \frac{360^o}{2\pi} = \frac{180^o}{\pi} \approx 57°$$

MEMORISE

Thus, 1 radian is the angle ≈ 57° such that whatever the radius, the corresponding circular arc has the same length as this radius – see figure 6.1. Similarly, for any angle,

$$\Rightarrow x \text{ (rad)} = x\frac{180^o}{\pi\text{(rad)}} = y^0$$

MEMORISE

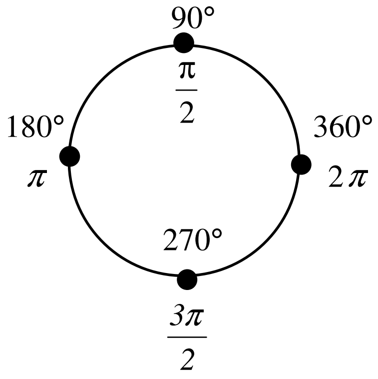
where x is the number of radians (angles which are ≈ 57°) in a given angle and y is the number of degrees in this angle.

Note: **The radian is a dimensionless (not arbitrary) unit** of angle.

Question: What is the angle $y = 90°$ in radians and why?
Answer:

Question: What is the angle $y = 180°$ in radians and why?
Answer:

Question: What is the angle $y = 270°$ in radians and why?
Answer:



MEMORISE

Here are a few more frequently used angles:

$30^0 = (\frac{30\pi}{180} =)\ \frac{\pi}{6}$ $60^0 = (\frac{60\pi}{180} =)\ \frac{\pi}{3}$ $120^0 = (\frac{120\pi}{180} =) \frac{2\pi}{3}$	$45^0 = (\frac{90\pi}{180} =) \frac{\pi}{4}$

MEMORISE

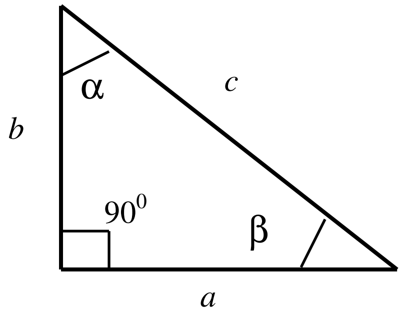
6.3.2 A historical note

"The concept of radian measure, as opposed to the degree of an angle, is normally credited to Roger Cotes in 1714. He had the radian in everything but name, and he recognized its naturalness as a unit of angular measure. The idea of measuring angles by the length of the arc was used already by other mathematicians. For example al-Kashi (c. 1400) used so-called diameter parts as units where one diameter part was 1/60 radian and they also used sexagesimal subunits of the diameter part.

The term radian first appeared in print on 5 June 1873, in examination questions set by James Thomson (brother of Lord Kelvin) at Queen's College, Belfast. He used the term as early as 1871, while in 1869, Thomas Muir, then of the University of St Andrews, vacillated between rad, radial and radian. In 1874, Muir adopted radian after a consultation with James Thomson." **http://en.wikipedia.org/wiki/Radian**

6.3.3 The Right Angle Triangles, Pythagoras Theorem and Trigonometric Ratios

1. The **Right Angle Triangle**



The **right angle** is the angle of 90^0. Side c which is opposite the right angle is called the **hypotenuse**.

Note: if $\alpha + \beta = 90^0$, then and α and β are called **complementary angles**.

2. **Pythagoras Theorem** states that in any right angle triangle

$$c^2 = a^2 + b^2$$

3. **The trigonometric ratios** $\sin$, $\cos$ and $\tan$ are defined for **acute angles** (that is, the angles that are less than 90^0) inside any right angle triangle as follows:

$$\sin\alpha = \frac{\text{length of side opposite to } \alpha}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\cos\alpha = \frac{\text{length of side adjacent to} \alpha}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\tan a = \frac{\text{length of side opposite to} \alpha}{\text{length of side adjacent to} \alpha} = \frac{a}{b}$$

Interestingly, these ratios do not depend on the length of the triangle sides. The above definitions imply two simple trigonometric **identities** (**formulae**, mathematical statements that are always true):

$$\sin\alpha = \cos\beta$$

$$\sin\beta = \cos\alpha \, .$$

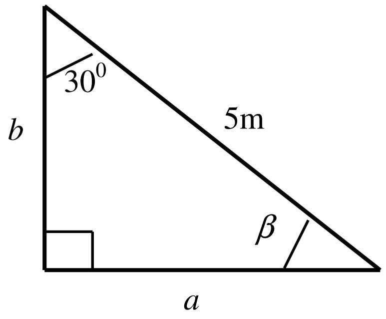
Here are frequently used specific trigonometric ratios:

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\tan\frac{\pi}{4} = 1$

MEMORISE

Examples:

1. Consider the following right-angle triangle:



a) $\beta = 90^0 - 30^0 = 60^0$

b) $a = \frac{1}{2}b$

$a^2 + b^2 = c^2$

$\frac{1}{4}b^2 + b^2 = c^2$

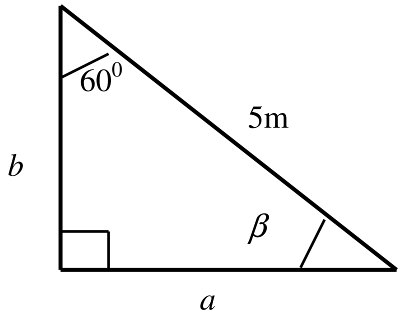
$\frac{5}{4}b^2 = 25$

$b^2 = 20$

$b = 2\sqrt{5}$

$a = \sqrt{5}$

2. Consider the following right-angle triangle:



a) $\beta = 90^0 - 30^0 = 60^0$

b) $a = 2b$

$a^2 + b^2 = c^2$

$4b^2 + b^2 = c^2$

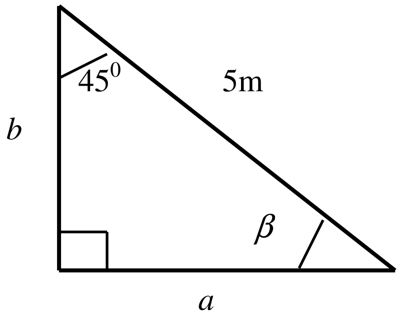
$5b^2 = 25$

$b^2 = 20$

$b = \sqrt{5}$

$a = 2\sqrt{5}$

3. Consider the following right-angle triangle:



a) $\beta = 90^0 - 45^0 = 45^0$

b) a = b

$a^2 + b^2 = c^2$

$2b^2 = c^2$

$2b^2 = 25$

$b^2 = \frac{25}{2}$

$b = \frac{5}{\sqrt{2}}$

$a = \frac{5}{\sqrt{2}}$

6.4 A historical note

”The method of logarithms was first publicly propounded in 1614, in a book entitled *Mirifici Logarithmorum Canonis Descriptio,* by John Napier, Baron of Merchiston, in Scotland. (Joost Bürgi independently discovered logarithms; however, he did not publish his discovery until four years after Napier. Early resistance to the use of logarithms was muted by Kepler's enthusiastic support and his publication of a clear and impeccable explanation of how they worked.

Their use contributed to the advance of science, and especially of astronomy, by making some difficult calculations possible. Prior to the advent of calculators and computers, they were used constantly in surveying, navigation, and other branches of practical mathematics. It supplanted the more involved method of prosthaphaeresis,… a quick method of computing products”. **http://en.wikipedia.org/wiki/Logarithm#History**

6.5 Instructions of self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise ALGEBRA Lecture 4 and study Solutions to Exercises in Lecture 4 using the STUDY SKILLS Appendix**
- **Revise FUNCTIONS Lecture 5 using the STUDY SKILLS Appendix**
- **Study Lecture 6 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Simplify using algebraic laws and rules if you can and then evaluate if you can:

a) $\frac{10^5}{10^3}$

b) $\frac{10^5}{2^3}$

c) $\frac{10^3}{5^5}$

d) 2^{3^2}

e) $\sqrt{x^2+a^2}$

f) $(\sqrt{x^2})^2$

g) $\ln e^{10}$

h) $\ln e^{100}$

i) $e^{\ln 100}$

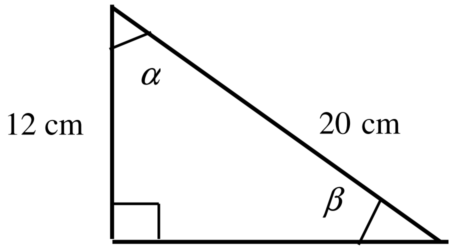
j) $e^{\log_{10} 100}$

Q2.

a) Sketch the function $f(x)=x^3$.

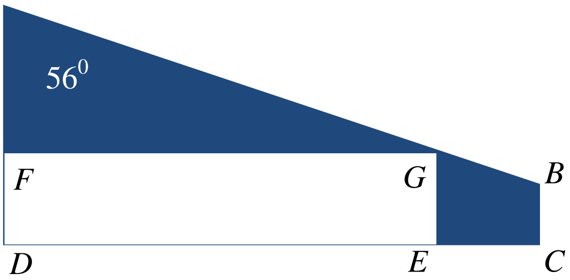
b) Sketch its inverse $f^{-1}(x)$.

Q3. a) Given the right angle triangle



find the length of the side opposite the angle α using Pythagoras Theorem.

b) $ABCD$ is the end wall in an attic. Segment AB is the sloping ceiling. AD = 1600 mm.



Ramona decides to hide a large gift box, $DEGF$, in the attic. If the box is 600 mm high, what is its length?

c) Express the following angles in radians:

i) 120^0
ii) 135^0
iii) 150^0
iv) 210^0
v) 240^0
vi) 300^0
vii) 330^0

d) Express the following angles in degrees:

i) $\frac{\pi}{5}$
ii) $\frac{5\pi}{12}$
iii) 3

Q4.

a) Prove that Pythagoras Theorem can be reformulated as $\sin^2\alpha + \cos^2\alpha = 1$.

b) Use Pythagoras Theorem to prove that

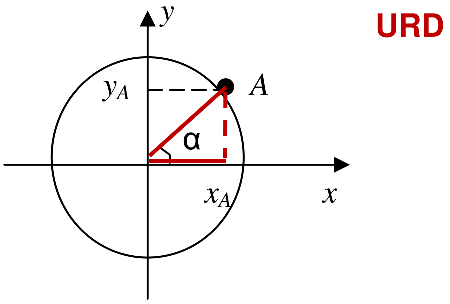
$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

c) Given that $\sin\frac{\pi}{6}=\frac{1}{2}$ use the Pythagoras Theorem to prove that $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$.

Lecture 7. Real FUNCTIONS of One Real Variable: Trigonometric Functions, Inverse Trigonometric Functions, Hyperbolic Functions

7.1 Trigonometric functions

We can generalise trigonometric ratios using **URD** (the **Unit Rod Diagram**) to define **trigonometric functions** of *any* angle, not necessarily acute. **URD** is a circumference of radius 1 supplied with the Cartesian coordinate system whose origin lies at the centre of the circumference.



Every point A on URD is described by its polar angle α. We project the point A onto the x-axis and y-axis.

Question: In the resulting right-angle triangle, what is the relationship between the length x_A of the side adjacent to α and point A?
Answer:

.
Question: What is the relationship between the length y_A of the side opposite to α and point A?
Answer:

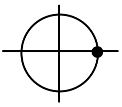
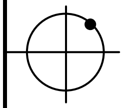
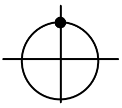
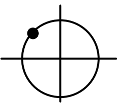
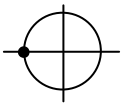
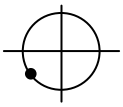
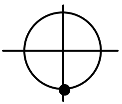
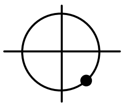
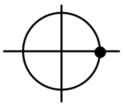
Question: What is $\cos\alpha$ here?
Answer:

Question: What is $\sin \alpha$ here?
Answer:

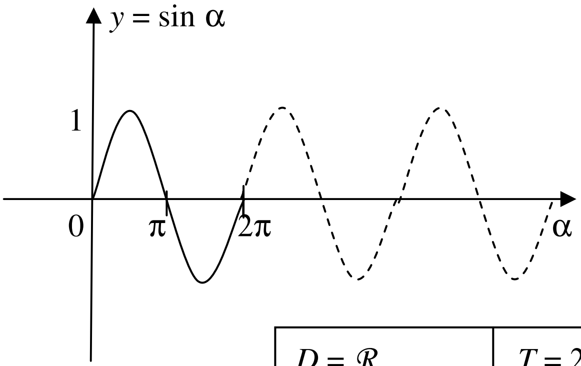
Generalisation: from now on, for any angle α, acute or not, we say that $\cos\alpha = x_A$ and $\sin\alpha = y_A$, where A is the point on the URD whose polar angle is α and the Cartesian coordinates are x_A and y_A.

We can sketch the graphs of the newly introduced functions $y = \sin\alpha$ and $y = \cos\alpha$ using the following table:

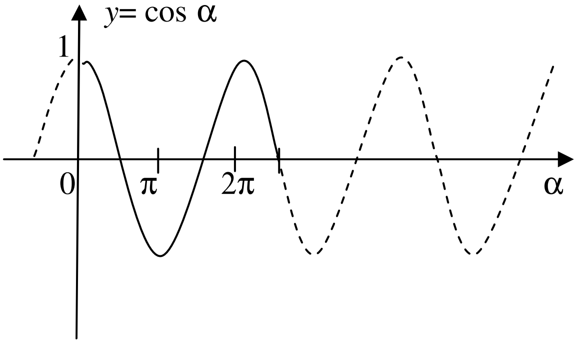
Table:

angle α	point A	cos α = x_A	sin α = y_A
0		1	0
$\frac{\pi}{4}$		$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{2}$		0	1
$\frac{3\pi}{4}$		$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
π		-1	0
$\frac{5\pi}{4}$		$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{3\pi}{2}$		0	-1
$\frac{7\pi}{4}$		$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
2π		1	0

Graphs of two functions, sin and cos:



$D = \mathbb{R}$ R: $-1 \le y \le 1$	$T = 2\pi$



The entry $T = 2\pi$ in the table directly underneath the graph of the sin-function means that both sin and cos functions are **periodic**, i.e. there exists a number T: $f(t + T) = f(t)$. The smallest positive T is called a **period**.

7.1.1 Useful Trigonometric Identities (Formulae)

1. $\sin x = \cos(\frac{\pi}{2} - x)$, $\quad\cos x = \sin(\frac{\pi}{2} - x)$,

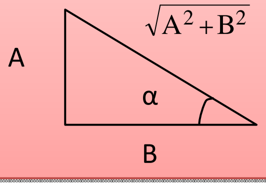
Note: this is true for acute angles x and $\frac{\pi}{2}$ - x. They are then complementary. Prove that this identity holds for all angles x and not just acute angles.

2. $\sin^2 x + \cos^2 x = 1$,
   $\sin(-x) = -\sin x$,
   $\cos(-x) = \cos x$,
   $\tan(-x) = -\tan x$, where $\tan x = \frac{\sin x}{\cos x}$.

Note: we can check these identities by using URD or else by using graphs of $\sin$ and $\cos$ functions.

Extra Trigonometric Identities

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x + \alpha), \text{ where } \tan\alpha = \frac{A}{B}$$



7.2 Inverse trigonometric functions

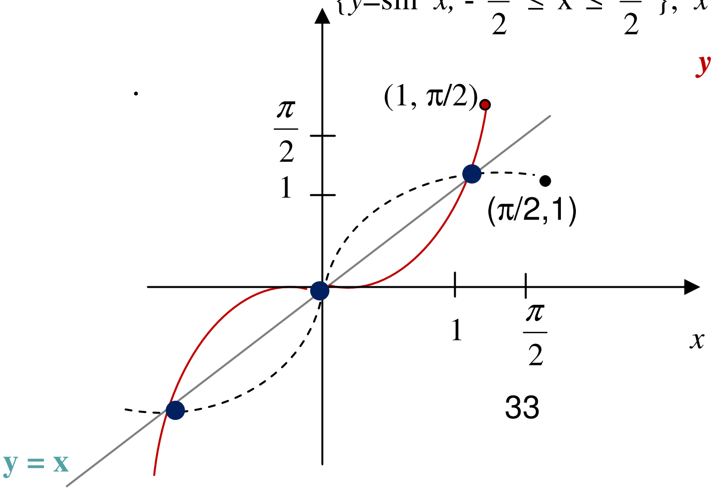
7.2.1 Inverse Sine ($\sin^{-1}$, arcsin)

$F(x) = \sin x$ has **no inverse on its natural domain**, because to every y we can assign any number of x's.

Let us **restrict its domain** to the interval where $\sin x$ is only **increasing** or only **decreasing** (goes up, up, up or down, down, down, respectively). Conventionally, we choose the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. When using notations $\sin^{-1}$ or arcsin this is the choice implied.

$\{y=\sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\}$, $x = \sin^{-1} y$ (dashed line),

$y = \sin^{-1} x$ (solid line)

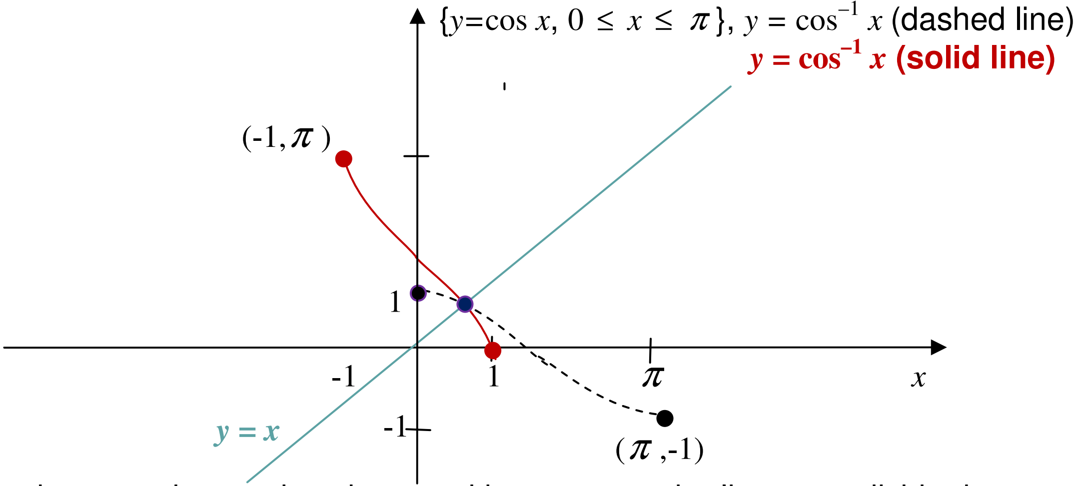


Note: when sketching the inverse of a function by creating its mirror image with respect to the line $y = x$ divide the graph of the original function into portions utilising special points. Then reflect the original graph portion by portion. In the case of sin-graph the special points are the intersections with the $y = x$ line. These points do not move when performing reflection (nothing changes when x and y exchange places since they are the same).

7.2.2 Inverse Cosine ($\cos^{-1}$, arcos)

$f(x) = \cos x$ has **no inverse on its natural domain**, because to every y we can assign any number of x's.

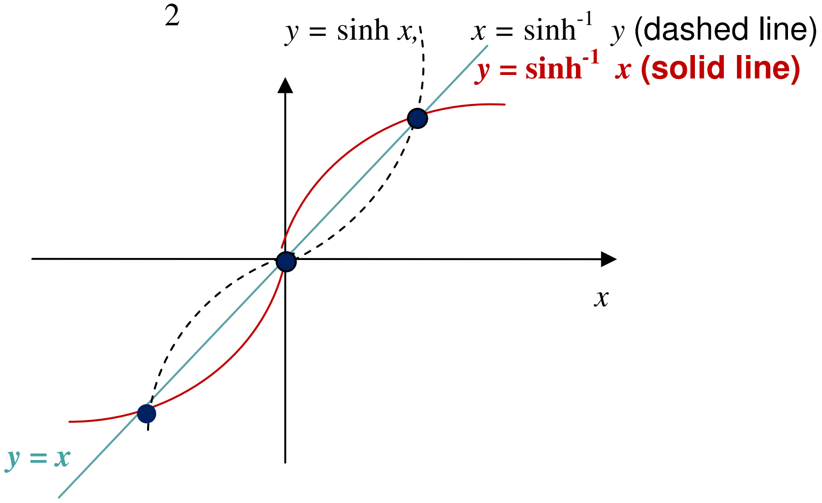
Let us **restrict its domain** to the interval where $\cos x$ is only increasing or only decreasing (goes up, up, up or down, down, down, respectively). Conventionally, we choose the interval $0 \le x \le \pi$. When using notations $\cos^{-1}$ or arccos this is the choice implied.



Note: when creating a mirror image with respect to the line $y = x$ divide the cos graph into portions by using its intersection with the $y = x$ line and its intersection with the x-axis). Then reflect the original graph portion by portion.

7.3 Hyperbolic functions and their inverses

7.3.1 The Hyperbolic Sine, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

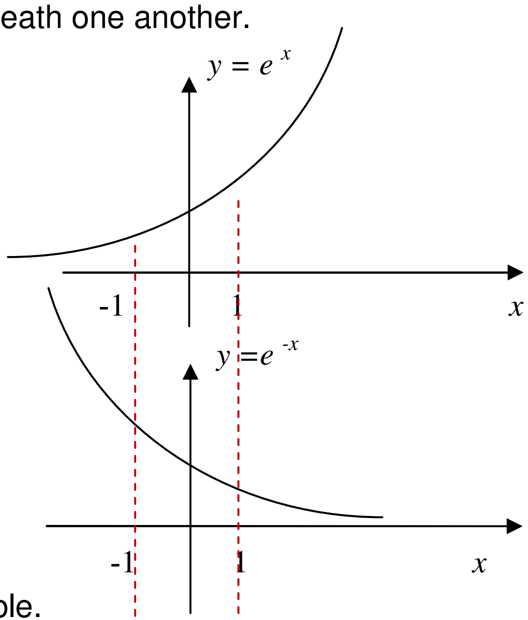


Note: The hyperbolic sin can be **sketched pointwise** (point by point), by using elementary operations on functions (see Section 4.3). Here are the steps to follow:

Step 1. Use tables to draw $y = \mathrm{e}^x$ and $y = \mathrm{e}^{-x}$ underneath one another.

x	0	1	-1	2	-2
e^x	1	$\mathrm{e} \approx 3$	$\frac{1}{e} \approx \frac{1}{3}$	$\mathrm{e}^2 \approx 9$	$\frac{1}{e^2} \approx \frac{1}{9}$

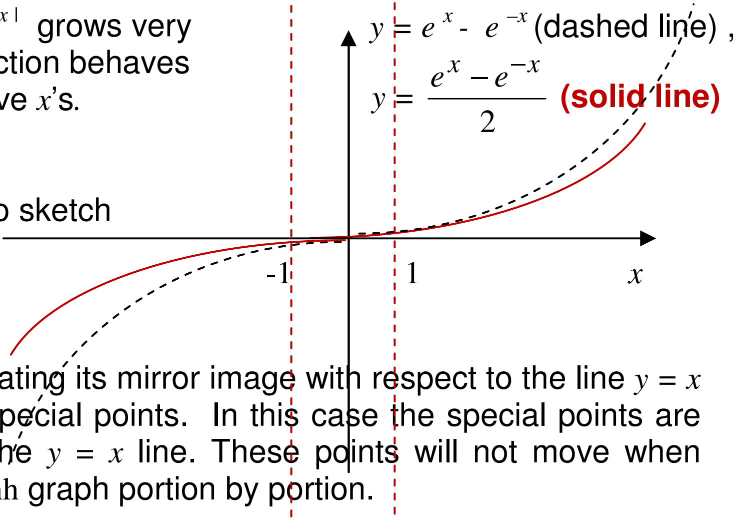
x	0	-1	1	-2	2
e^{-x}	1	$e \approx 3$	$\frac{1}{e} \approx \frac{1}{3}$	$e^2 \approx 9$	$\frac{1}{e^2} \approx \frac{1}{9}$



Step 2. Use pointwise subtraction to sketch $y = \mathrm{e}^{x} - \mathrm{e}^{-x}$. Use the same x-values as in the table. Then note that when $|x|$ grows large $\mathrm{e}^{-|x|}$ grows very small, so that when $|x|$ is large the function behaves as e^x for positive x's and as $-\mathrm{e}^{-x}$ for negative x's.

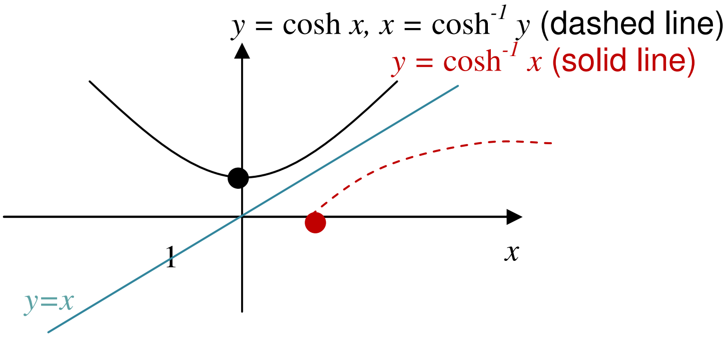
Step 3. Use the pointwise division by 2 to sketch

$$y = \frac{e^x - e^{-x}}{2}.$$



Note: When sketching the inverse by creating its mirror image with respect to the line $y = x$ divide the graph into portions by using special points. In this case the special points are the intersection of the sinh graph with the $y = x$ line. These points will not move when performing reflection. Then reflect the sinh graph portion by portion.

7.3.2 The Hyperbolic Cosine, $\cosh x = \dfrac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}$



The hyperbolic $\cosh$ can be **sketched pointwise**, by using elementary operations on functions as above – first addition, then division by 2 (see Section 4.3).

The hyperbolic function $f(x) = \cosh x$ has **no inverse on its natural domain**, because to every y we can assign two x's. Let us **restrict its domain** to the interval where $\cosh x$ is only increasing or only decreasing (goes up, up, up or down, down, down, respectively). Conventionally, $y = \cosh^{-1} x$ is inverse function to $y = \cosh x, x \geq 0$

7.4 Instructions for self-study

- **Revise Summaries on FUNCTIONS and ORDER OF OPERATIONS**
- **Revise Lecture 5 and study Solutions to Exercises in Lecture 5 using the STUDY SKILLS Appendix**
- **Revise Lecture 6 using the STUDY SKILLS Appendix**
- **Study Lecture 7 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1.

a) Sketch $y = \tan x$.
b) What is its natural domain?
c) What is its period?
d) Sketch $y = \tan^{-1} x$.

Q2. Simplify

a) $y = \sin(-x)$
b) $y = \sin(\frac{\pi}{2} - x)$
c) $y = \cos(-x)$
d) $y = \cos(\frac{\pi}{2} - x)$

Q3.

a) Explain why function $y = x^2$ has no inverse.
b) Suggest how many inverses can you define for it?
c) Sketch all of them.

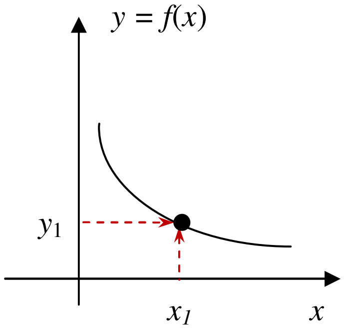
Q4.

a) Sketch the function $y = \tanh x$.
b) Sketch its inverse.

Lecture 8. Real FUNCTIONS of One Real Variable: Sketching and Using Graphs, Simple Transformations

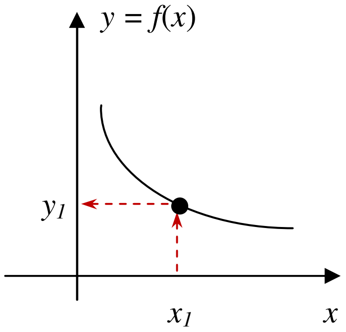
In Lecture 5 we have learned how to sketch graphs using a table as well as how to use these graphs to find a value of a dependent variable given a value of an independent variable and *vice versa*. Here is some revision.

8.1 Sketching graphs using a table



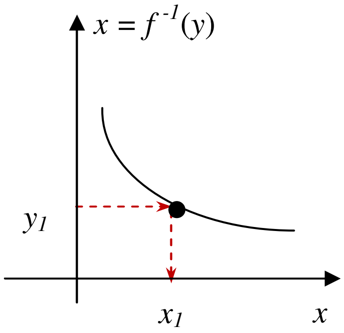
Verbalise: Given a value x_1 of independent variable and a value y_1 of dependent variable find the point with x-coordinate x_1 and y-coordinate y_1.

8.2 Using graphs to find y_1 given x_1



Verbalise: Given a value x_1 of independent variable find a point on the curve with x-coordinate x_1 and project it onto the y-axis to find the value y_1 of dependent variable.

8.3 Using graphs to find x_1 given y_1



Verbalise: Given a value y_1 of dependent variable find the point on the curve with y-coordinate y_1 and project it onto the x-axis to find the value x_1 of dependent variable

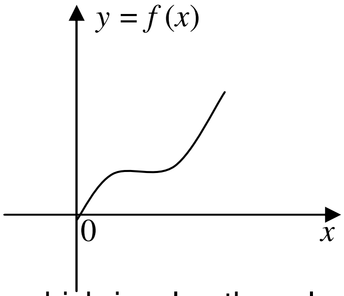
Note: the same graph can be used to visualise function $y = f(x)$ and its inverse $x = f^{-1}(y)$ – **provided this inverse exists.**

8.4 Sketching graphs using simple transformations

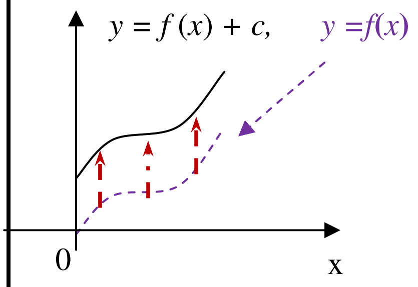
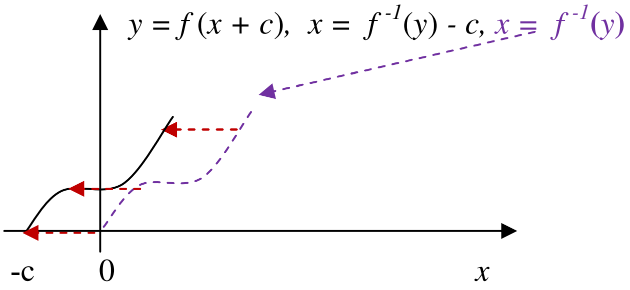
Sketching graphs using the table is advisable when trying to recall the shape of one of elementary functions. However, if you do not know the shape of the graph of a composite function you are likely to create an incomplete table. There are more reliable ways of sketching composite functions.

Question: Which elementary functions have we covered?
Answer:

Let us now assume that we know how to sketch a function $y = f(x)$. Say it looks like that



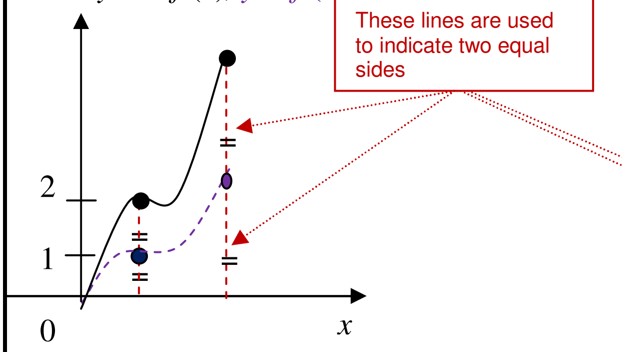
Let us consider composite functions which involve the rule $f(\ )$ and addition of a constant to an independent variable x or dependent variable y, or else multiplication by a constant of an independent variable x or dependent variable y. Such composite functions can be sketched using **simple transformations** (translation, scaling or reflection) of the curve $y = f(x)$. **In order to see what these transformations are**, it is useful to make **the transformed variable the subject of the equation**. Let us discuss the six relevant transformations in detail: (below we use the abbreviation **wrt** for *with respect to*)

Sketch $y = f(x) + c$ ($+c$ – **last operation**)	Sketch $y = f(x + c)$ ($+c$ – **first operation**)
Let $c > 0$  **Algebraic description:** addition of a constant c to y **For every x,** old $y = f(x)$ transformed $y = f(x) + c \Rightarrow$ **transformed** y = **old** $y + c$ **Geometrical description:** translation wrt y-axis by c $c > 0$ - shifting up, $c < 0$ - shifting down	Let $c > 0$  **Algebraic description:** addition of a constant c to x **For every y,** old $x = f^{-1}(y)$ transformed $x = f^{-1}(y) - c \Rightarrow$ **transformed** x = **old** $x - c$ **Geometrical description:** translation wrt x-axis by $-c$ $c > 0$ - shifting left, $c < 0$ - shifting right

Sketch $y = cf(x)$, $c > 0$($\cdot c$ – **last operation**)

Let $c = 2$

$y = 2f(x)$, $y = f(x)$



Algebraic description:
multiplication of y by a +ve constant c

For every x, old $y = f(x)$
transformed $y = c \cdot f(x) \Rightarrow$

transformed $y = c \cdot$ **old** y

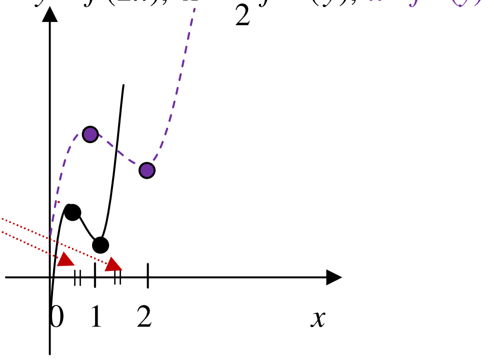
Geometrical description:
scaling wrt y-axis by c

$c > 1$ - stretching wrt y-axis
$c < 1$ - squashing wrt y-axis

Sketch $y = cf(x)$, $c > 0$ ($\cdot c$ – **first operation**)

Let $c = 2$

$y = f(2x)$, $\mathrm{x} = \frac{1}{2} f^{-1}(y)$, $x = f^{-1}(y)$



Algebraic description:
multiplication of x by +ve constant c

For every y, old $x = f^{-1}(y)$
transformed $x = \frac{1}{c} f^{-1}(y) \Rightarrow$

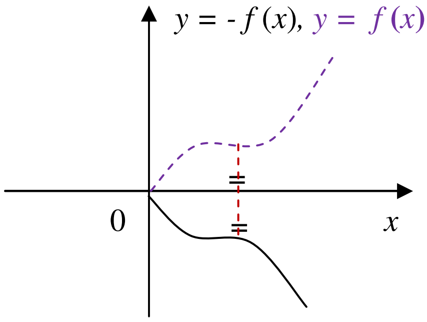
transformed $x = \frac{1}{c} \cdot$ **old** x

Geometrical description:
scaling wrt y-axis by $\frac{1}{c}$

$c > 1$ - squashing wrt x-axis
$c < 1$ - stretching wrt x-axis

Sketch $y = -f(x)$ ($\cdot(-1)$ – **last operation**)

$y = -f(x)$, $y = f(x)$



Algebraic description:
multiplication of y by -1

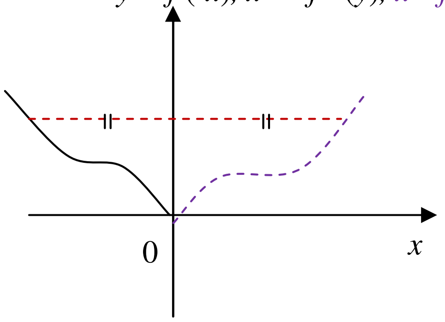
For every x, old $y = f(x)$, new $y = -f(x) \Rightarrow$

transformed $y = -$ **old** y

Geometrical description:
reflection wrt the x-axis

Sketch $y = f(-x)$ ($\cdot(-1)$ – **first operation**)

$y = f(-x)$, $x = -f^{-1}(y)$, $x = f^{-1}(y)$



Algebraic description:
multiplication of x by -1

For every y, old $x = f^{-1}(y)$,
transformed $\mathrm{x} = -f^{-1}(y) \Rightarrow$

transformed x =- old x

Geometrical description:
reflection wrt the y-axis

8.5 Sketching graphs using several simple transformations

If a composite function involves several operations of addition of, or multiplication by, a constant, then it can be sketched by simple transformations using the following recipe:

Step 1. Drop all constant factors and terms.

Step 2. Bring the constants back one by one in **Order of Operations** (not necessary but advisable) and at each step use the **Decision Tree** in figure 8.1 to decide which simple transformation is affected by each constant. Sketch the resulting graphs underneath one another.

Note 1: if c is a **negative factor**, write $c = (-1) * |c|$, so it affects two operations.

Note 2: if c affects **neither the first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations first.

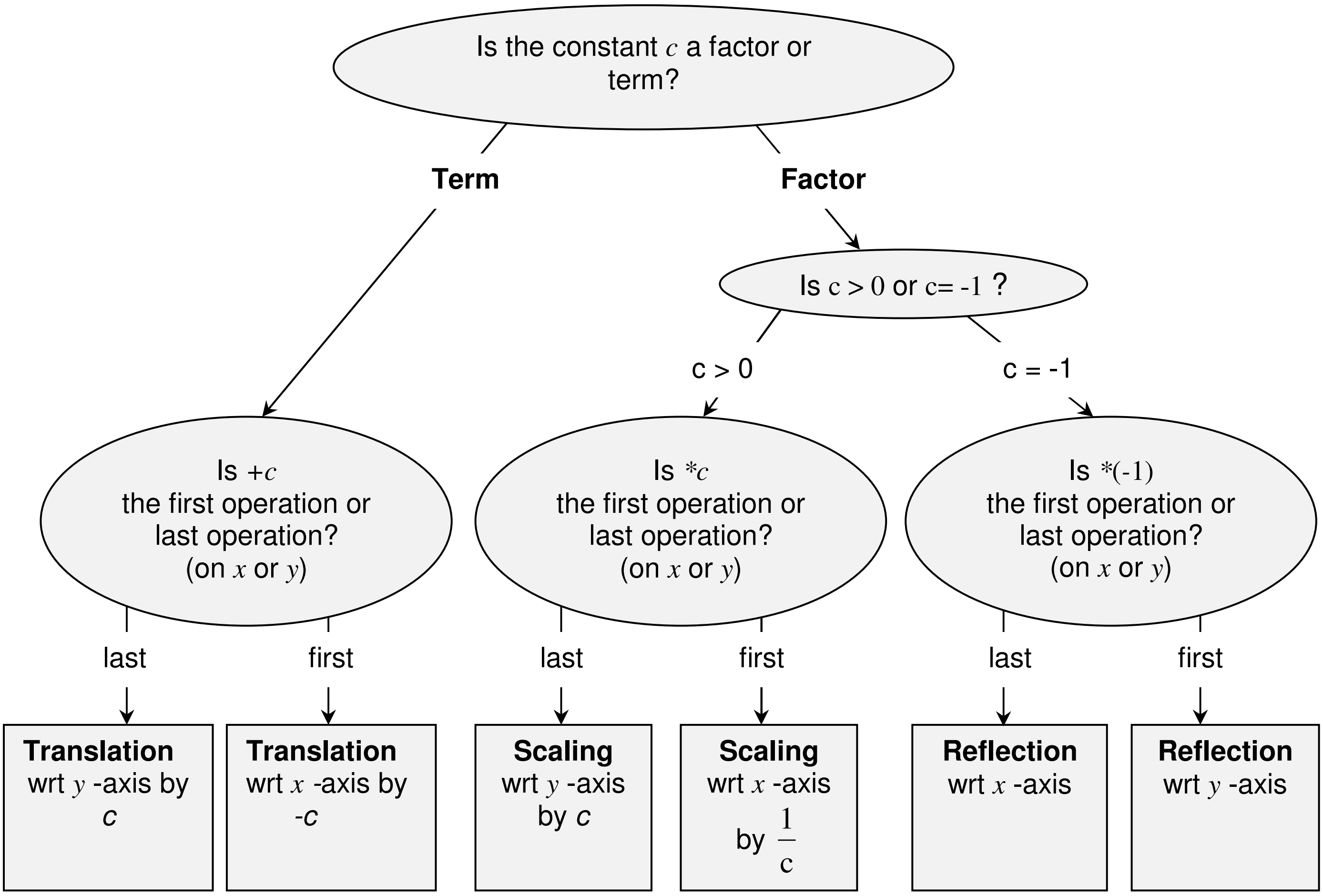


Figure 8.1. Decision Tree for Sketching by Simple Transformations which are due to multiplication by, or addition of, a constant c to x or y in $y = f(x)$.

Note 3: If **the same operation** is applied to all x's or y's - as for example in $y = \ln(x - 2) + \sin(x - 2)$ - this operation should be treated as one operation.

Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is affected by a constant c multiplying y or added to y, y should be made the subject of the equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\frac{1}{c}$, respectively.

8.6 Completing the square

In Section 3.2.1 we have introduced a trick called completing the square.

Completing the square means re-writing the quadratic expression

$$ax^2 + bx + c = (ax^2 + bx) + c = a(x^2 + \frac{b}{a}x) + c = a[(x + \frac{1}{2}\frac{b}{a})^2 - (\frac{1}{2}\frac{b}{a})^2] + c =$$

$a(x + \frac{1}{2}\frac{b}{a})^2 - \frac{1}{4}\frac{b^2}{a} + c$, so that the unknown is present only in one – square - term.

It is useful for sketching a parabola by simple transformations.

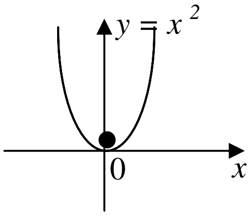
Example: Sketch $y = x^2 + 6x + 4$ by simple transformations.

Solution

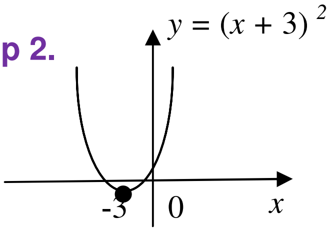
$y = x^2 + 6x + 4 = (x + 3)^2 - 9 + 4 = (x + 3)^2 - 5$

Drop all constant factors and terms and then bring them back in **Order of Operations**.

Step1.

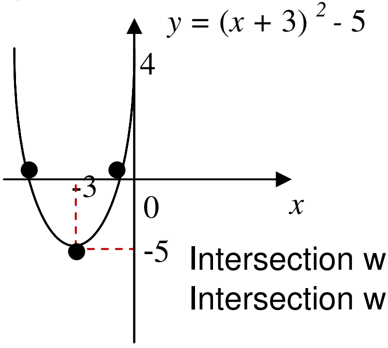


Step 2.



$+3$ – constant term, first operation
$\Rightarrow$ translation wrt x-axis by -3

Step 3.



$+5$ – constant term, last operation
$\Rightarrow$translation wrt y-axis by -5

Intersection with the x-axis: $y = 0$, $x_{1,2} = -6 \pm \sqrt{9-4} \approx -3 \pm 2 =$ -1, -5
Intersection with the y-axis: $x = 0$, $y = 4$

8.7 Applications of simple transformations

To give just one example, any linear dynamic system (any linear electrical, mechanical or chemical device) transforms the input signal (which is a real function of time) into the output signal (which is a real function of time) by affecting multiplication by constants and addition of constants. In particular, if the input is sinusoidal the linear transformation involves scaling of the amplitude and shifting of the phase.

8.8 Instructions for self-study

- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 6 and study Solutions to Exercises in Lecture 6 (sketching elementary and inverse functions) using the STUDY SKILLS Appendix**
- **Revise Lecture 7 using the STUDY SKILLS Appendix**
- **Study Lecture 8 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Sketch using simple transformations:

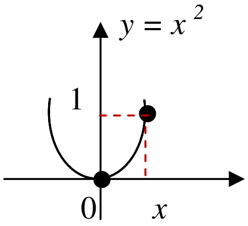
a) $y = \sin 2x$
b) $y = 2 \sin x$
c) $y = \cos (\frac{1}{2} x)$
d) $y = \frac{1}{2} \cos x$
e) $y = \mathrm{e}^{3x}$
f) $y = 3\,\mathrm{e}^{x}$
g) $y = \ln (\frac{1}{2} x)$
h) $y = \frac{1}{2} \ln x$
i) $y = \cos x + 2$
j) $y = \cos x - 2$
k) $y = \sin (x + 2)$
l) $y = \sin (x - 2)$
m) $y = \mathrm{e}^{x+3}$
n) $y = \mathrm{e}^{x-3}$
o) $y = \frac{1}{2}\mathrm{x}^2 + 2x - 3$ (complete the square first).

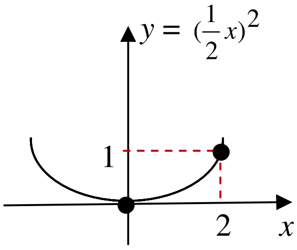
Lecture 9. Real FUNCTIONS of One Real Variable: Sketching Graphs by Simple Transformations (ctd.)

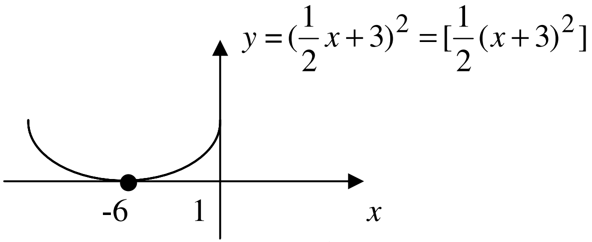
9.1 Sketching graphs using several simple transformations (ctd.)

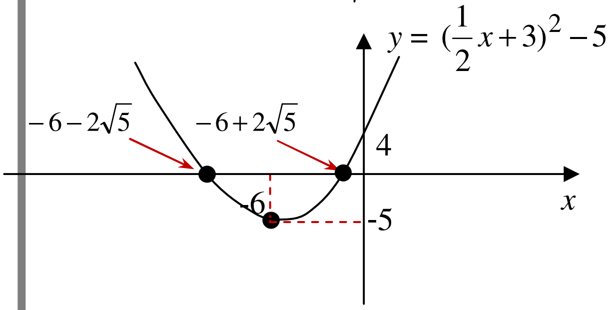
Example: Sketch the parabola $y=(\frac{1}{2}x+3)^2-5$

Solution









Intersection with the x-axis: $y=0$, $(\frac{1}{2}x+3)^2-5=0$, $x_{1,2}=-6\pm2\sqrt{5}$.

Intersection with the y-axis: $x=0$, $y=4$.

9.2 Sketching graphs using pointwise operations

Here is a recipe for sketching graphs using pointwise operations:

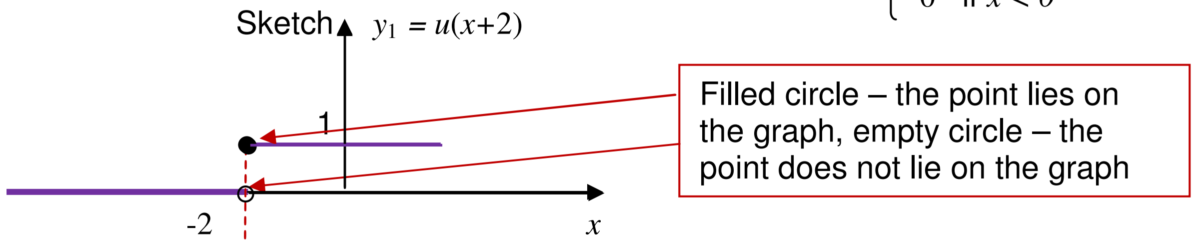
Step 1. Decompose the function into simpler ones starting with the last operation.
Step 2. Sketch each simple function separately (you might need to decompose it as well).
Step 3. Compose the original function by pointwise operations.

Example: $y = u(x+2)[\,(\frac{1}{2}x+3)^2-5\,]$.

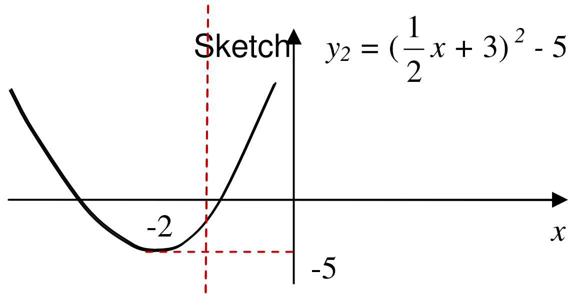
Solution

Step 1. Decompose $y(x)$ as $y(x)= y_1(x)\, y_2(x)$, where $y_1(x) = u(x+2)$, $y_2(x) = [\,(\frac{1}{2}x+3)^2-5\,]$

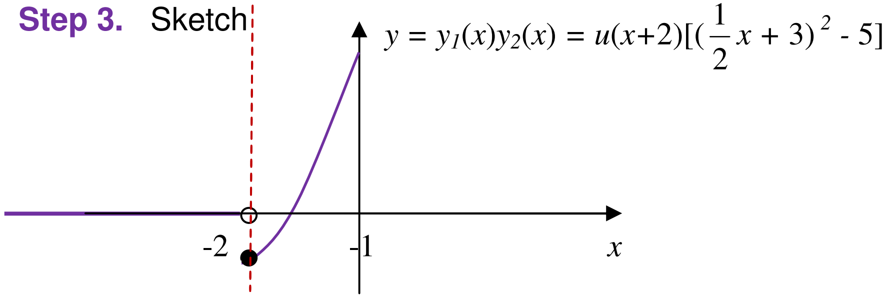
Step 2. Sketch y_1 and y_2, where the unit step function $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$



Filled circle – the point lies on the graph, empty circle – the point does not lie on the graph



Step 3. Sketch



9.3 Instructions for self-study

- **Revise ALGEBRA Summary (addition and multiplication)**
- **Revise Summaries on ORDER OF OPERATIONS and FUNCTIONS**
- **Study Lecture 7 and study Solutions to Exercises in Lecture 7 using the STUDY SKILLS Appendix**
- **Study Lecture 8 using the STUDY SKILLS Appendix**
- **Study Lecture 9 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Sketch

a) $y = (2x + 3)^2 - 4$

b) $y = 2\,e^{-x}$

c) $y = \ln(\frac{x}{2}-3)^{\frac{1}{2}} = \frac{1}{2}\ln(\frac{1}{2}x-3)$

d) $y = \sin(\frac{\pi}{2} - x)$

e) $y = u(x - 6)\,[\,(\frac{1}{2}x + 3)^2 - 5\,] + u(x - 3)(\,x^2 + 6x + 4)$, where $u(x)$ is the unit step.

f) $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Lecture 10. ALGEBRA: Addition of Complex Numbers, the Argand Diagram, Forms of Complex Numbers

In Elementary Algebra we study **variables** and **operations on variables**. As discussed in Lectures 1 - 3, **inverse algebraic operations** introduce new types of numbers:

- subtraction introduces **negative** numbers,
- division introduces **rational** numbers,
- roots and logs introduce **irrational** and **complex numbers.**

10.1 Imaginary unity *j*

Let us introduce a new number j,

$$j \equiv \sqrt{-1}: \quad j^2 = -1 \tag{10.1}$$

It is easy to check that j cannot be real.

Proof:
Assume that $\sqrt{-1} = x$, where x is real. Then squaring both sides, $-1 = x^2$. However, any real number squared is positive or zero. Hence the **assumption** (that there exists a real number x such $\sqrt{-1} = x$) **is wrong.**

j is a new type of number called the **imaginary unity.**

10.2 Applications of complex numbers: solving quadratic equations

Consider the quadratic equation

$$ax^2 + bx + c = 0, \tag{10.2}$$

Using Section 3.1.6, if $b^2 – 4ac < 0$ we can write

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{(-1)(4ac - b^2)}}{2a} \frac{-b \pm j\sqrt{4ac - b^2}}{2a}. \tag{10.3}$$

Note: every quadratic equation has exactly two roots, but sometimes they coincide and sometimes they are not real.

Example: $x^2 + x + 1 = 0$, where $a = 1,\ b = 1,\ c = 1$

$$x_{1,2} = \frac{-b \pm j\sqrt{4ac - b^2}}{2} = \frac{-1 \pm j\sqrt{4-1}}{2} = \frac{-1}{2} \pm j\frac{\sqrt{3}}{2}$$

10.3 Operations: the whole powers of *j*

$$j = \sqrt{-1}, \quad j^2 = -1, \quad j^3 = j^2 \cdot j = -j, \quad j^4 = j^2 \cdot j^2 = 1,$$
$$j^5 = j^4 \cdot j = j, \quad j^6 = j^4 \cdot j^2 = -1, \quad j^7 = j^4 \cdot j^3 = -j, \quad j^8 = j^4 \cdot j^4 = 1, \ etc. \tag{10.4}$$

10.4 Variables: definition of a complex number z

In Section 10.2, Eq. (10.3) we introduced a **complex number** presented as $z = x + jy$, that is, as a sum of two terms, where one term, x is real and another term, jy is **purely imaginary** (that is, j times a real number y).

x is called the **real part** of z, $\mathrm{Re}\,(z)$.
y is called (illogically!) the **imaginary part** of z, $\mathrm{Im}\,(z)$.

Operations on complex numbers are performed the same way they are performed on any algebraic expression (we just apply the laws and rules of Algebra formally).

10.5 Operations: addition of complex numbers z_1 and z_2

Consider two complex numbers, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$. To represent their sum in the same form we just collect the like terms, that is, collect real terms separately and imaginary terms separately:

$$z_1 + z_2 = x_1 + x_2 + j\,(y_1 + y_2)$$

Examples: $(2 + 3j) + (-3 - 2j) = (2 - 3) + (3 - 2)\,j = -1 + j$
$(2 + 3j) + (-3 + 2j) = (2 - 3) + (3 + 2)\,j = -1 + 5j$

Rules of addition (check them yourself using the Laws of Addition of real numbers)

$z_1 + z_2 = z_2 + z_1$
$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

10.6 Operations: subtraction of complex numbers

The difference between two complex numbers is introduced in a standard way:

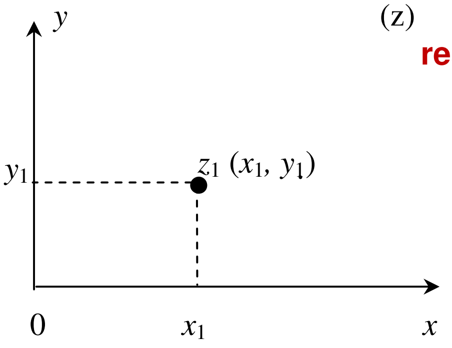
Definition: $z_1 - z_2 \equiv z_3 : z_3 + z_2 = z_1$
$\Rightarrow \quad z_1 - z_2 = x_1 - x_2 + j\,(y_1 - y_2)$

this symbol means **"implies"**

Examples: $(2 + 3j) - (3 + 2j) = (2\text{-}3) + (3\text{-}2)\,j = -1 + j$
$(2 + 3j) - (-3 + 2\mathrm{j}) = (2 + 3) + (3\text{-}2)\,j = 5 + j$

The above rules suggest that a complex number can be thought of as a pair of two real numbers. Since each real number can be represented **(visualised)** as a point on a number line, a complex number can be represented **(visualised)** using two number lines (for simplicity, let them be perpendicular to each other and let number 0 be represented on both number lines by the point of their intersection.) In other words, a complex number can be represented **(visualised)** as a point on a number plane otherwise known as **the complex plane** or **the Argand diagram**.

10.7 The Argand diagram and Cartesian form of a complex number



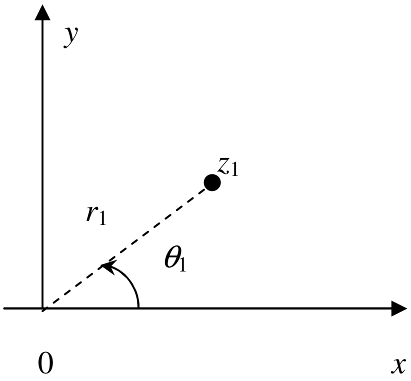
real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z = x + jy$, where x, y - real

x-coordinate y-coordinate   (on the ordinary plane)

Verbalise: In the **Cartesian representation,** a complex number is a sum of two terms, one real (x) and one purely imaginary (jy).

10.8 The Argand diagram and polar form of a complex number



magnitude, $|z|$ **argument,** arg(z)   (on the Argand diagram)

$z = r\angle\theta$, where $r \geq 0$, θ - angle
(in degrees or radians)

polar radius r polar angle θ (on the ordinary plane)
(distance to the origin)

Verbalise: in the **polar representation**, a complex number is a pair of polar coordinates r and θ rather than a pair of Cartesian coordinates x and y.

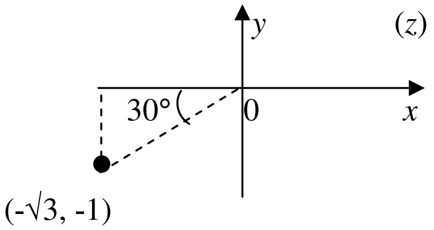
10.9 The Cartesian-polar and polar-Cartesian transformations

Using the above Argand diagram, Pythagoras Theorem and trigonometric ratios we can always transform Cartesian coordinates into polar coordinates and *vice versa*, thus converting one form of a complex number into another:

Cartesian →polar	**polar → Cartesian**
$r = \sqrt{x^2 + y^2}$ (Pythagoras Theorem) $\theta = \begin{cases} \tan^{-1}\dfrac{y}{x} & \text{if } x > 0 \\ \tan^{-1}\dfrac{y}{x} + \pi & \text{if } x < 0 \\ \operatorname{sgn}(y)\dfrac{\pi}{2} & \text{if } x = 0 \end{cases}$ where sgn(y) is a sign of y, + or -	$x = r\cos\theta$ $y = r\sin\theta$

Examples:

1. What is the polar representation of $z = -\sqrt{3} - 1j$? Use the Argand diagram.

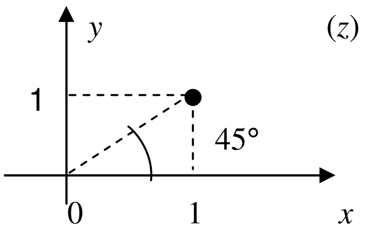


$$r = \sqrt{3+1^2} = 2$$

$$\theta = \tan^{-1}\frac{1}{\sqrt{3}} + 180° = 30° + 180° = 210°$$

$$z = 2 \underline{\left| 210° \right.}$$

2. What is the Cartesian representation of $z = 1 \underline{\left| 45° \right.}$? Use the Argand diagram.



$$x = 1 \cdot \cos 45° = \frac{\sqrt{2}}{2}$$

$$y = 1 \cdot \sin 45° = \frac{\sqrt{2}}{2}$$

$$z = \frac{\sqrt{2}}{2} + j\,\frac{\sqrt{2}}{2}$$

10.10 The trigonometric form of a complex number

Using the Cartesian - polar transformation we can represent a complex number in the so-called **trigonometric form**:

$$z = x + jy = r(\cos\theta + j\sin\theta)$$

10.11 Euler's formula

There is a remarkable formula that connects the sin-function, cos-function and the exponential function:

Euler's formula: $\cos\theta + j\sin\theta = e^{j\theta}$

MEMORISE

It can be proven later using the Maclaurin series.

10.12 The exponential form of a complex number

Using the Euler's formula, the trigonometric representation can be re-written in the so-called **exponential form**:

$z = r\,e^{j\theta}$, $r \geq 0$ or, using the Argand diagram notation, $z = |z|\,e^{j\arg(z)}$,

where $\theta = \arg(z)$ is real (dimensionless number, angle in radians)

MEMORISE

Verbalise: In the exponential representation, a complex number is a product of two factors, one positive (r) and one $(e^{j\theta})$ - an imaginary exponent (that is, e to power which is purely imaginary, j times a real number).

Examples:

1. Express $z = 1\,\underline{\lfloor 45^\circ}$ in the exponential form. **Answer:** $z = e^{j\pi/4}$

2. Express $z = 2\,\underline{\lfloor 210}^\circ$ in the exponential form. **Answer:** $z = 2e^{j7\pi/6}$

3. Express $z = \frac{\sqrt{2}}{2}(1 + j)$ in the exponential form. **Answer:** using the Argand diagram,

$$r = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1,\ \theta = \tan^{-1} 1 = \frac{\pi}{4},\ z = e^{j\pi/4}$$

10.13 Applications

“In electrical engineering, the… treatment of resistors, capacitors, and inductors can … be unified by introducing imaginary, frequency-dependent resistances for the latter two and combining all three in a single complex number called the impedance. This use is also extended into digital signal processing and digital image processing… to transmit, compress, restore, and otherwise process digital audio signals, still images, and video signals.”
http://en.wikipedia.org/wiki/Complex_number

10.14 A historical note

“The earliest fleeting reference to square roots of negative numbers perhaps occurred in the work of the Greek mathematician and inventor Heron of Alexandria in the 1st century AD, when he considered the volume of an impossible frustum of a pyramid, though negative numbers were not conceived in the Hellenistic world.

Complex numbers became more prominent in the 16th century, when closed formulas for the roots of cubic and quartic polynomials were discovered by Italian mathematicians… Niccolo Fontana Tartaglia, Gerolamo Cardano…. It was soon realized that these formulas, even if one was only interested in real solutions, sometimes required the manipulation of square roots of negative numbers.”
http://en.wikipedia.org/wiki/Complex_number#History

10.15 Instructions for self-study

- **Revise ALGEBRA Summary (addition and subtraction)**
- **Revise Summaries on QUADRATICS and TRIGONOMETRY**
- **Revise Lecture 8 and study Solutions to Exercises in Lecture 8 using the STUDY SKILLS Appendix**
- **Revise Lecture 9 using the STUDY SKILLS Appendix**

- **Revise Lecture 10 using the STUDY SKILLS Appendix**
- **Study Section 4 the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Solve the quadratic equation $5x^2 - 11x + 13 = 0$

b) Given $z_1 = -1 + 2j$ and $z_2 = -1 - 6j$ find

i) $z_1 + z_2$

ii) $z_1 - z_2$

iii) $z_1 - 2z_2$

iv) $2z_1 + z_2$

Q2. Write down the real and imaginary parts of

a) $3 + 7j$

b) $-0.35j$

c) $\cos \omega t + j \sin \omega t$, where ω and t are real

d) $\mathrm{jV} \sin(\omega t + \varphi)$, where V, ω, t and φ are real

Q3. Show the following complex numbers on the Argand diagram

a) $3e^{j\pi/3}$

b) $\sqrt{2}e^{j2\pi}$

c) $3e^{-j\pi}$

d) $5e^{j0}$

Q4. Find the real and imaginary parts of

a) $4e^{j\pi/3}$

b) $e^{j\pi/2}$

c) $3e^{-j\pi/2}$

d) e^{j-1}

Q5. Find $|z|$ and $\arg(z)$ and express z in the exponential form for

a) $z = -1 + 2j$

b) $z = -3 - 6j$

Lecture 11. ALGEBRA: Multiplication and Division of Complex Numbers

11.1 Multiplication of a complex number z by a real number α

Given a complex number $z = x + jy = r\,e^{j\theta}$ and a real number α, using the *Cartesian form* we can write

$$\alpha z = \alpha x + \alpha j y$$

and using the *exponential form*, we have

$$\alpha z = \begin{cases} \alpha r e^{j\theta} & \text{if } \alpha \ge 0 \\ \alpha r e^{j(\theta+\pi)} & \text{if } \alpha < 0 \end{cases}$$

Examples:

1. Given $z = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$ and $\alpha = \sqrt{2}$ find αz. **Answer:** $\alpha z = 1 + j$.

2. Given $z = \frac{1}{\sqrt{2}}(\cos\theta + j\sin\theta)$ and $\alpha = \sqrt{2}$ find αz. **Answer:** $\alpha z = \cos\theta + j\sin\theta$.

3. Given $z = e^{j\pi/4}$ and $\alpha = \sqrt{2}$ find αz. **Answer:** $\alpha z = \sqrt{2}\, e^{j\pi/4}$.

11.2 Operations: multiplication of complex numbers z_1 and z_2

Using the *Cartesian form* we can write

$$z_1 \cdot z_2 = (x_1 + jy_1)(x_2 + jy_2) = x_1 x_2 - y_1 y_2 + j\,(y_1 x_2 + x_1 y_2)$$

Multiplication is easier to perform in the *exponential form:*

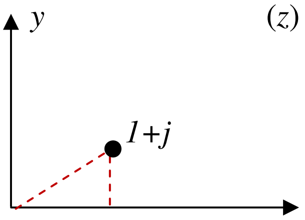
$$z_1 \cdot z_2 = (r_1 r_2) e^{j(\theta_1+\theta_2)}$$

Examples:

1. Find $(1 + j)(1 - j)$ in the Cartesian form.

Solution

$(1 + j)(1 - j) = 1 + 1 + j\,(1 - 1) = 2$



2. Find $(1+j)(1-j)$ in the exponential form.

Solution

$$1+j=\sqrt{2}e^{j\frac{\pi}{4}}\,(\tan^{-1}\frac{1}{1}=\frac{\pi}{4})$$

$$1-j=\sqrt{2}e^{-j\frac{\pi}{4}}\,(\tan^{-1}\frac{1}{1}=\frac{\pi}{4})$$

$$(1+j)(1-j) = 2\,e^{j(\pi/4-\pi/4)} = 2$$

3. $j\cdot z = e^{j\pi/2}\,r e^{j\theta} = r e^{j(\theta+\pi/2)}$

⇒ multiplication by *j* is equivalent to rotation on the Argand diagram by 90° anti-clockwise.

11.3 Operations: complex conjugate z^* of a complex number z

Using the *Cartesian form* we define the complex conjugate of z as $z^* = x - jy$. Some authors use a bar instead of the star.

It follows that in the *exponential form* the complex conjugate of z is $z^* = r e^{-j\theta}$.

Examples:

In the *Cartesian form*, $z\,z^* = x^2 + y^2$.
In the *exponential form*, $z\,z^* = |r|\,e^{j\theta}\,|r|\,e^{-j\theta} = r^2\,e^{j\theta - j\theta} = r^2\,e^0 = r^2 \times 1$.
$\Rightarrow z\,z^* = r^2$.

11.4 Operations: division of complex numbers z_1 and z_2

In the *Cartesian form,* we can use the complex conjugate of the denominator and represent the ratio of two complex numbers as a sum of a real and imaginary term,

$$\frac{z_1}{z_2} = \frac{\overset{x_2 - jy_2}{x_1 + jy_1}}{x_2 + jy_2} = \frac{x_1x_2 + y_1y_2 + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

Division is much easier to perform in the *exponential form*:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,e^{j(\theta_1 - \theta_2)}.$$

Example: Find $\frac{1+j}{1-j}$.

Solution

In the *Cartesian form,* we have

$$\frac{1+j}{1-j}\overset{1+j}{\diagdown} = \frac{0+2j}{2} = j.$$

In the *exponential form*, we have

$$\frac{1+j}{1-j} = \frac{\sqrt{2}e^{j\pi/4}}{\sqrt{2}e^{-j\pi/4}} = e^{j\pi/2} = j.$$

11.5 Integer powers of complex numbers

De Moivre's Theorem

$$z_n = r^n(\cos n\theta + j\sin n\theta)$$

where r – magnitude of z, |z|, θ - argument of z, $\arg(z)$, and n – integer.

Proof: **using Euler's formula**

$$z^n = (r\,e^{j\theta})^n = r^n\,(e^{j\theta})^n = r^n\,e^{jn\theta} \overset{\downarrow}{=} r^n(\cos n\theta + j\sin n\theta)$$

Examples:

1. $(1+j)^2 = (\sqrt{2}\,e^{j\pi/4})^2 = 2\,e^{j\pi/2} = 2j$
   (use the Argand diagram to put $1+j$ into the exponential form first)

2. $(1-j)^2 = (\sqrt{2}\,e^{-j\pi/4})^2 = 2e^{-j\pi/2} = -2j$
   (use the Argand diagram to put $1+j$ into the exponential form first)

11.6 The fractional powers $\frac{1}{k}$ (*k-th* roots) of complex numbers

$e^{j2\pi n} = 1$, where n is an integer
(check by putting 1 into the exponential form).

$$\Rightarrow e^{j\theta} = e^{j\theta} \times 1 = e^{j\theta} \times e^{j2\pi n} = e^{j(\theta + 2\pi n)}$$

Thus, any complex number can be represented as

$$z = r e^{j\theta} = r e^{j(\theta + 2\pi n)},\ n - \text{integer},\ -\pi < \theta_0 \le \pi.$$

Verbalise: $2\pi n$ is n full rotations around the origin – clockwise or anticlockwise – of the point z**.** These rotations do not change the point's position on the Argand diagram, and therefore, adding $2\pi n$ to the argument of a complex number does not change z **.**

Example:

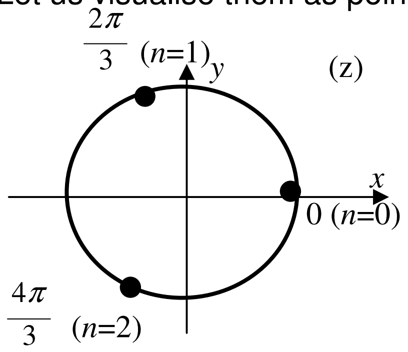
1. Find $1^{1/3}$.

Solution

$1 = e^{2\pi n j}$, n – integer

$1^{1/3} = (e^{2\pi n j})^{1/3} = e^{j\frac{2\pi}{3}n}$.

These roots are complex numbers in the exponential form.
Let us visualise them as points on the Argand diagram:



$$\begin{array}{llll} n = 0 \Rightarrow \theta = 0, & n = 3 & \theta = 2\pi \\ n = 1 \Rightarrow \theta = \frac{2\pi}{3}, & n = 4 & \theta = \frac{8\pi}{3} \\ n = 2 \Rightarrow \theta = \frac{4\pi}{3}, & n = 5 & \theta = \frac{10\pi}{3} \end{array}$$

whatever other n, the corresponding number is represented on the Argand diagram by one of the above three points.

Answer: 3 roots, $\frac{2\pi}{3}$ (*rad*)**, that is,** 120^0 **apart.**

11.7 Instructions for self-study

- **Revise ALGEBRA Summary (powers and roots)**
- **Revise Summaries on TRIGONOMETRY and COMPLEX NUMBERS**
- **Revise Lecture 9 and study Solutions to Lecture 9 using the STUDY SKILLS Appendix**

- **Revise Lecture 10 using the STUDY SKILLS Appendix**
- **Study Lecture11 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find $(1+2j)(1-j)$

b) Find $(1+2j)(1-j)(3j-1)$

c) Find $z_1 z_2$, where $z_1 = \frac{1}{3}e^{j\pi/3}, z_2 = 3e^{j\pi/3}$

d) Show that multiplying any complex number by $-j$ means rotating the corresponding point on the Argand diagram around the origin through 90^0 clockwise.

Q2.

a) Find $\frac{1+j}{1-2j}$

b) Find $\frac{z_1}{z_2}$ where $z_1 = \frac{1}{3}e^{j\pi/3},\ z_2 = 3e^{-j\pi/3}$

c) Find $\frac{1}{j}$

d) Show $j^* = \frac{1}{j}$

Q3.

a) Use De Moivre's Theorem to prove $\cos 2x = \cos^2 x - \sin^2 x$

b) Use De Moivre's Theorem to prove $\sin 2x = 2\sin x \cos x$

c) Find $(3+j4)^{1/2}$

d) Solve $z^2 + 2z + 1 - j = 0$

Additional Exercises

1. Find $z_1 z_2$, where $z_1 = 3-2j,\ z_2 = 4-3j$
2. Find $(3-2j)j(4-3j)$
3. Find $z_1 z_2$, where $z_1 = \frac{1}{\sqrt{3}}e^{j\pi/6}, z_2 = \sqrt{3}e^{-j\pi/3}$
4. Find $\frac{1+j}{2-j}$
5. Find $\frac{z_1}{z_2}$ where $z_1 = \frac{1}{\sqrt{3}}e^{j\pi/6}, z_2 = \sqrt{3}e^{-j\pi/3}$
6. Solve $-z^2 + 3z - 4 = 0$
7. Solve $z^3 = -1$

Lecture 12. ALGEBRA: Fractional Powers, Logs and Loci of Complex Numbers

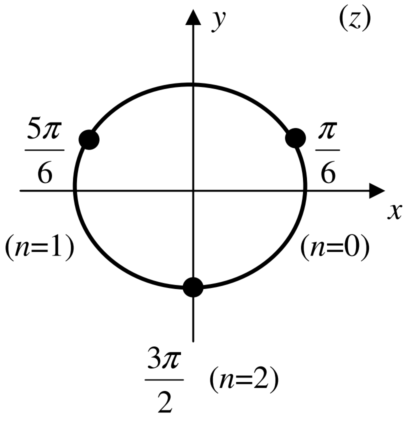
12.1 The fractional powers $\frac{1}{k}$ (whole roots k) of complex numbers (ctd.)

Examples:

1. Find $j^{1/3}$.

Solution

Step 1. Represent j in the exponential form: $j = e^{j(\pi/2 + 2\pi n)}$, n - integer
Step 2. Raise j to the fractional power 1/3: $j^{1/3} = [e^{j(\pi/2 + 2\pi n)}]^{1/3} = e^{j(\pi/6 + 2\pi/3n)}$
Step 3. Visualise all roots as points on the Argand diagram:



$n = 0 \qquad \theta = \frac{\pi}{6}$

$n = 1 \qquad \theta = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$

$n = 2 \qquad \theta = \frac{\pi}{6} + \frac{4\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}$

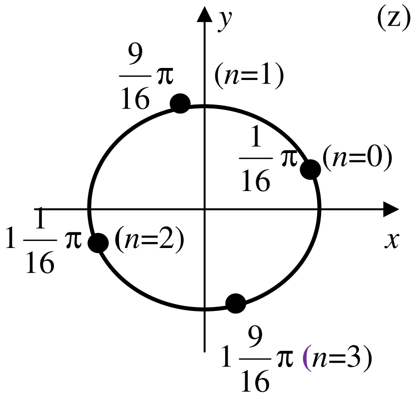
2. Find $(1 + j)^{1/4}$.

Solution

Step 1. Represent $1+j$ in the exponential form: $1+j = \sqrt{2}\, e^{j\pi/4} = \sqrt{2}\, e^{j(\pi/4+2\pi n)}$, n - integer
Step 2. Raise $1+j$ to the fractional power 1/4:

$(1+j)^{¼} = [\sqrt{2}e^{j(\pi/4 + 2\pi n)}]^{1/4} = 2^{1/8}\, e^{j[\pi/16 + (2\pi/4)n]}$

Step 3. Visualise all roots as points on the Argand diagram:



$n = 0 \qquad \theta = \frac{\pi}{16}$

$n = 1 \qquad \theta = \frac{\pi}{16} + \frac{2\pi}{4} = \frac{9\pi}{16}$

$n = 2 \qquad \theta = \frac{\pi}{16} + \frac{2\pi \cdot 2}{4} = \frac{17\pi}{16} = 1\frac{1}{16}\pi$

$n = 3 \qquad \theta = \frac{\pi}{16} + \frac{2\pi \cdot 3}{4} = \frac{25\pi}{16} = 1\frac{9}{16}\pi$

- 4 roots, $\pi/2$ (*rad*) $= 90^0$ **apart**

12.2 Logs of complex numbers

Using the exponential form of a complex number z,

$$\ln z \;=\; \ln\,(r\,e^{j(\theta_0 + 2\pi n)}) \;=\; \ln r \;+\; j\theta_0 \;+\; j2\pi n, \quad n - \text{integer.}$$

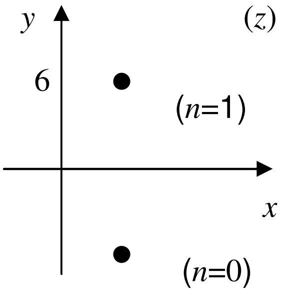
Example: Find $\ln\,(1 - j)$.

Solution

Step 1. $1 - j = \sqrt{2}\, e^{-j\pi/4}$

Step 2. $\ln\,(1 - j) = \ln\,(\sqrt{2}\, e^{-j\pi/4}) = \ln\sqrt{2} + \ln e^{-j(\pi/4 + 2\pi n)} = \ln\sqrt{2} - j\dfrac{\pi}{4} + j2\pi n\,,\; n \;=\; \pm 1, \pm 2, \ldots$

Step 3. Visualise them as points on the Argand diagram:



12.3 Loci on the Argand diagram

Locus is a geometrical location of points with a specific property.

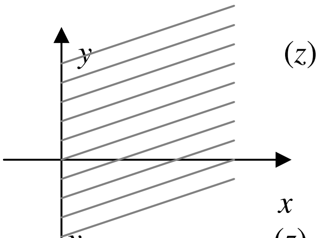
Remember: $z \;=\; \cos\,\theta \;+\; j\,\sin\,\theta \;=\; r\,e^{j\theta}$

Let us consider a few examples of the type you will often encounter in many of your future engineering courses.

1. On the Argand diagram, the locus of points with property

$$\operatorname{Re}(z) \geq 0$$

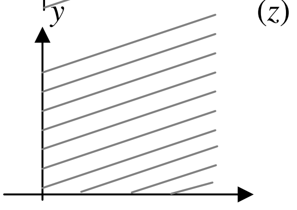
is the right half-plane.



2. On the Argand diagram, the locus of points with property

$$0 \leq \arg(z) \leq \frac{\pi}{2}$$

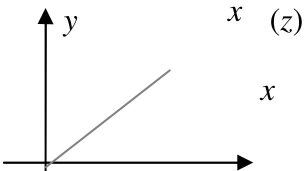
is a quarter plane which is the 1st quadrant.



3. On the Argand diagram, the locus of points with property

$$\arg(z) = \frac{\pi}{4}$$

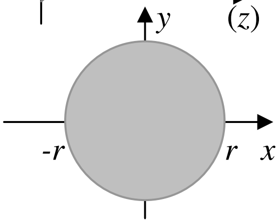
is a ray issuing from the origin at 45^0 to the horizontal axis.



4. On the Argand diagram, the locus of points with property

$$|z| = r$$

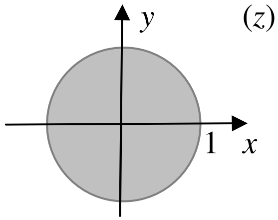
is the circumference of radius of r centred at the origin.



5. On the Argand diagram, the locus of points with property

$$x^2 + y^2 \leq 1$$

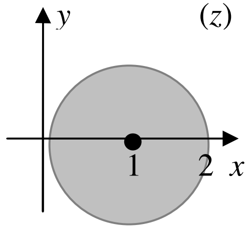
is the circle of radius of 1 centred at the origin.



6. On the Argand diagram, the locus of points with property

$$(x-1)^2 + y^2 \leq 1$$

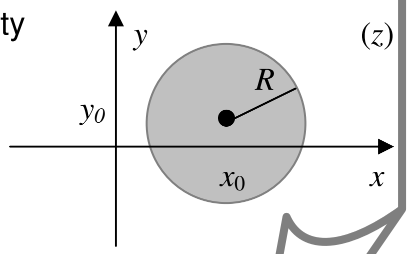
is the circle of radius of 1 centred at (1,0).



7. On the Argand diagram, the locus of points with property

$$(x - x_0)^2 + (y - y_0)^2 \leq R^2$$

is the circle of radius of R centred at (x_0, y_0).



12.4 Applications

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. A transfer function is a ratio of two polynomials in a complex variable s. The locus of zeros of these polynomials is a useful tool for analysing *stability* of SISOs. They are stable if all zeros of the denominator are in the left-hand side of the complex s-plane.

12.5 Instructions for self-study

- **Revise ALGEBRA Summary (powers and logs)**
- **Revise Summaries on TRIGONOMETRY and COMPLEX NUMBERS**
- **Revise Section 10 and study Solutions to Exercises 10 using the STUDY SKILLS Appendix**
- **Revise Section 11 the STUDY SKILLS Appendix**
- **Study Section 12 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. a) Find the cubic roots of $z = \frac{(-3+j)^4}{(2-j)^2}$

b) Solve the equation $z^4 + 25 = 0$.

Q2. Assuming z – complex and x, y – real, indicate on the Argand diagram the following loci:

a) $\mathrm{Re}(z) \le 2$

b) $(x-1)^2 + (y-2)^2 = 2$

c) $\left|\frac{1}{1+jy}\right| < 1$ **(advanced)**

d) $\arg(\frac{1}{1+jy}) = 1$**(advanced).**

III. SUMMARIES

Algebra Summary

OPERATIONS

Addition (direct operation)

Addition of whole numbers gives whole number

1. $a + b = b + a$

Terminology: a and b are called **terms**
$a + b$ is called **sum**

2. $(a + b) + c = a + (b + c)$

Subtraction (inverse operation)

Def : $a - b = x$**:** $x + b = a$

Note: $a + b - b = a$ **(subtraction undoes addition)**
$a - b + b = a$ **(addition undoes subtraction)**

3. $a + 0 = a$

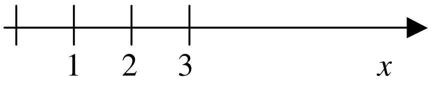
4. for each $\underline{a}$ there exists one **additive inverse** $-a$:
$a + (-a) = 0$

Rules (follow form Laws):

$+(b + c) = +b + c$
$+ \ a + b = a + b$
$-(-a) = a$
$-(a) = -a$

TYPES OF VARIABLES

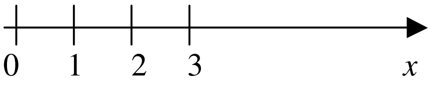
Whole numbers are $1, 2, 3, \ldots$



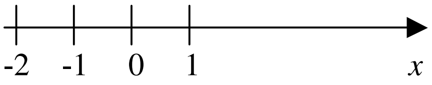
introduces **0** and **negative numbers**:

$a - a = 0$
if $b > a$ $\quad a - b = -(b - a)$

Natural numbers are $0, 1, 2, \ldots$



Integers are ..., -2, -1, 0, 1, 2, ...



Multiplication (direct operation)

For whole numbers n

$$a\cdot n = \underbrace{a+\ldots\ldots\ldots\ldots\ldots+a}_{n\text{ times}}$$

Notation: $ab = a\cdot b = a\times b$

$2b = 2\cdot b = 2\times b$

$23 \neq 2\cdot 3,\ 23 = 2\cdot 10+3$

$2\frac{1}{2} \neq 2\cdot\frac{1}{2},\ 2\frac{1}{2} = 2+\frac{1}{2}$

$2\frac{3}{2} = 2\cdot\frac{3}{2}$

1. $a\cdot b = a\cdot b$

Terminology: a and b are called **factors**

ab - **product**

2. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

Conventions: $abc = (ab)c$

$a(-bc) = -abc$

3. $a(b+c) = ab+ac$

$\longrightarrow$ Removing brackets

$\longleftarrow$ Factoring

4. $a\cdot 0 = 0$

5. $a\cdot 1 = a$

Rules (follow from Laws):

$(a + b)(c + d) = ac + ad + bc + bd$ (**SMILE RULE**)

$(-1)\cdot n = -n$

$(-1)\cdot(-1) = 1$

Division (inverse operation)

Def: $a/b = x:\ \ xb = a$

Terminology: a – **numerator**

b – **denominator**

a/b – **fraction (ratio)**

proper fraction if $|a|<|b|$, a, b integers

Note: $ab/b = a$ **(division undoes multiplication)**

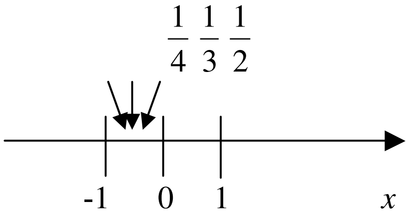
$(a/b)b = a$ **(multiplication undoes division)**

6. For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $a\cdot 1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\frac{m}{n}$,

where m and $n \neq 0$ are integers (**division by zero is not defined**)



Rules:

$$\frac{a}{b}\cdot n = \frac{an}{b}$$

$$\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$$

$$\frac{an}{bn} = \frac{a\cdot\cancel{n}}{b\cdot\cancel{n}} = \frac{a}{b}$$

CANCELLATION

$$\frac{1}{\frac{n}{m}} = \frac{m}{n}$$

FLIP RULE

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

Note: $\frac{a+c}{b} = (a+c)/b$

$$\frac{\overset{d}{\bcancel{a}}}{b}+\frac{\overset{b}{\bcancel{c}}}{d}=\frac{ad}{bd}+\frac{cb}{db}=\frac{ad+cb}{bd}$$

COMMON DENOMINATOR RULE

n-th **power** b^n (direct operation)
If n – a whole number

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

Rules

$a^m \cdot a^n = a^{m+n}$

(product of powers with the same base is a power with indices added)

$a^n \cdot b^n = (ab)^n$

(product of powers is power of product)

$a^m / a^n = a^{m-n}$

(ratio of powers is power of ratios)

$a^m / b^m = (a/b)^m$

(ratio of powers with the same base – subtract indices)

$a^0 = 1$

$a^{-n} = 1/a^n$

$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

***n-th* root** (inverse to taking to power n)

Def: $^n\sqrt{b} = x$: $x^n = b$

Note:

$\sqrt[n]{b^n} = b$

(taking *n-th* root undoes taking *n-th* power)

$(\sqrt[n]{b})^n = b$

(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = {}^n\sqrt{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

Logarithm base b (inverse to taking b to power)

Def: $\log_b a = n$: $b^n = a$

Note:

$\log_b b^n = n$ (check using definition: $b^n = b^n$)

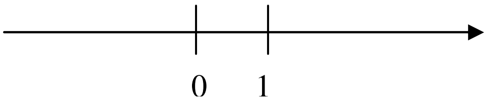
(taking $\log_b$ undoes taking b to power)

$b^{\log_b n} = n$

(taking b to power undoes $\log_b$)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, …

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole real line



introduces **irrational** (not rational**)** **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers,** $\sqrt{-1}, \log_{10}(-1)$, *etc.*

Rules (follow from Rules for Indeces):

$\log_b xy = \log_b x + \log_b y$

(log of a product is sum of logs)

$\log_b x/y = \log_b x - \log_b y$

(log of a ratio is difference of logs)

$\log_b 1 = 0$ **(log of 1 is 0)**

$\log_b b = 1$ **($b^1 = b$)**

$\log_b 1/a = -\log_b a$

$\log_b x^n = n \log_b x$

(log of a power is power times log)

$\log_b a = \log_c a/\log_c b$

(changing base)

General remarks

1. $a - b = a + (-b)$ ⟶ a difference can be re-written as a sum

2. $\frac{a}{b} = a \cdot \frac{1}{\mathrm{b}} = ab^{-1}$ ⟶ a ratio can be re-written as a product

3. ${}^n\sqrt{b} = b^{1/n}$ ⟶ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

6. By convention, in any algebraic expression operations should be performed using the Order of Operations convention.

Functions Summary

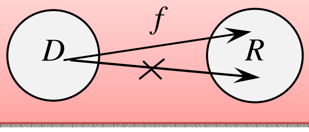
Variables are denoted mostly by $x, y, z, p, \ldots\ldots w$.
A variable can take any value from an set of allowed numbers.

Functions are denoted mostly by f, g and h or $f(\,)$, $g(\,)$ and $h(\,)$
(no multiplication sign is intended!).

In mathematics, the word **function** has two meanings:

1) $f()$ - an operation or a chain of operations on an **independent variable (argument)*;***
2) $f(x)$ – a **dependent variable** (a variable dependent on x**)**, that is the result of applying the operation $f(\,)$ to an independent variable x.

A diagrammatical representation of a function



To specify a function we need to specify a (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, $f(x)$ assigns one and only one value $y \in R$ (**range**, the set of all possible values of the dependent variable).

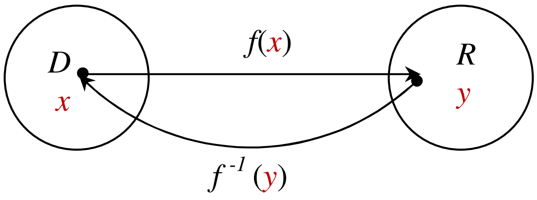
Inverse functions

$f^{-1}(x)$**:** $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function



$y = f(x)$

$x = f^{1}(y)$

Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. The **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[(   )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**, and some authors do not bracket **arguments** of elementary **functions**, such as exp, log, sin, cos, tan *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.*

Other **operations** must be performed in the order of decreasing complexity, which is

oiB - **operations in Brackets**
F - Functions $f(\ )$
P- Powers (including inverse operations of roots and logs)
M - Multiplication (including inverse operation of division)
A - Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of operations (OOO)

1. Make implicit (invisible) brackets visible (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO

oiB	f( )	P	M	A
(including		$(\ )^{(\ )}$	×	+
implicit)		roots	÷	-
		logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2+bx+c\,,$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared): b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2+bx+c=0\,.$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2+bx+c=a(x-x_1)(x-x_2).$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be 360° or 2π (radians)

$$\Rightarrow 2\pi\ (rad) = 360°$$
$$\Rightarrow 1\ (rad) \approx 57°$$

Above, x is the number of **radians** (angles ≈ 57°) in a given angle (*cf*: x (m) $= x\cdot$ 1m, so that x is the number of **units of length** (segments 1m long) in a given segment)

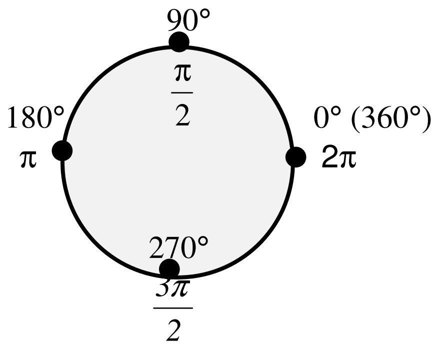
(**Note**: ***cf.*** means ***compare***)

The radian is a dimensionless unit of angle.

$$\Rightarrow\ x\ (rad) = x\frac{180^o}{\pi(\text{rad})} = y^0,\ \ y^0\ =\ y\frac{\pi(rad)}{180^0} = x\ (rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

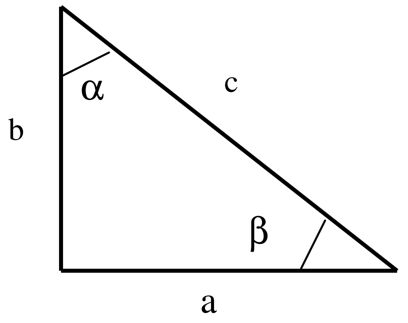
Frequently used angles



$30° = (\frac{30\pi}{180} =)\ \frac{\pi}{6}$	$45° = \frac{\pi}{4}$
$60° = (\frac{60\pi}{180} =)\ \frac{\pi}{3}$	
$120° = (\frac{120\pi}{180} =)\ \frac{2\pi}{3}$	

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90^0) as follows:



$$\sin\alpha = \cos\beta = \frac{a}{c}$$

$$\cos\alpha = \sin\beta = \frac{b}{c}$$

$$\tan a = \cot\beta = \frac{a}{b}$$

$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Trigonometric identities

$\sin^2 x + \cos^2 x = 1$

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

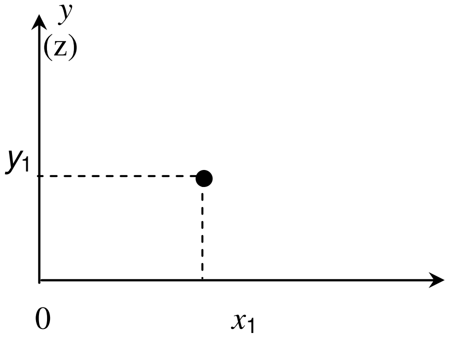
$\cos 2x = \cos^2 x - \sin^2 x$

$\sin 2x = 2\sin x \cos x$

$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x + \alpha)$, where $\tan\alpha = \frac{A}{B}$

Complex Numbers

The Cartesian representation of a complex number

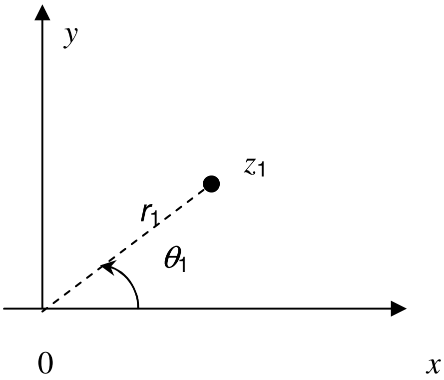


real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z= x+jy$, where x,y – real, $j = \sqrt{-1}$

x-coordinate y-coordinate (on the

The exponential representation of a complex number

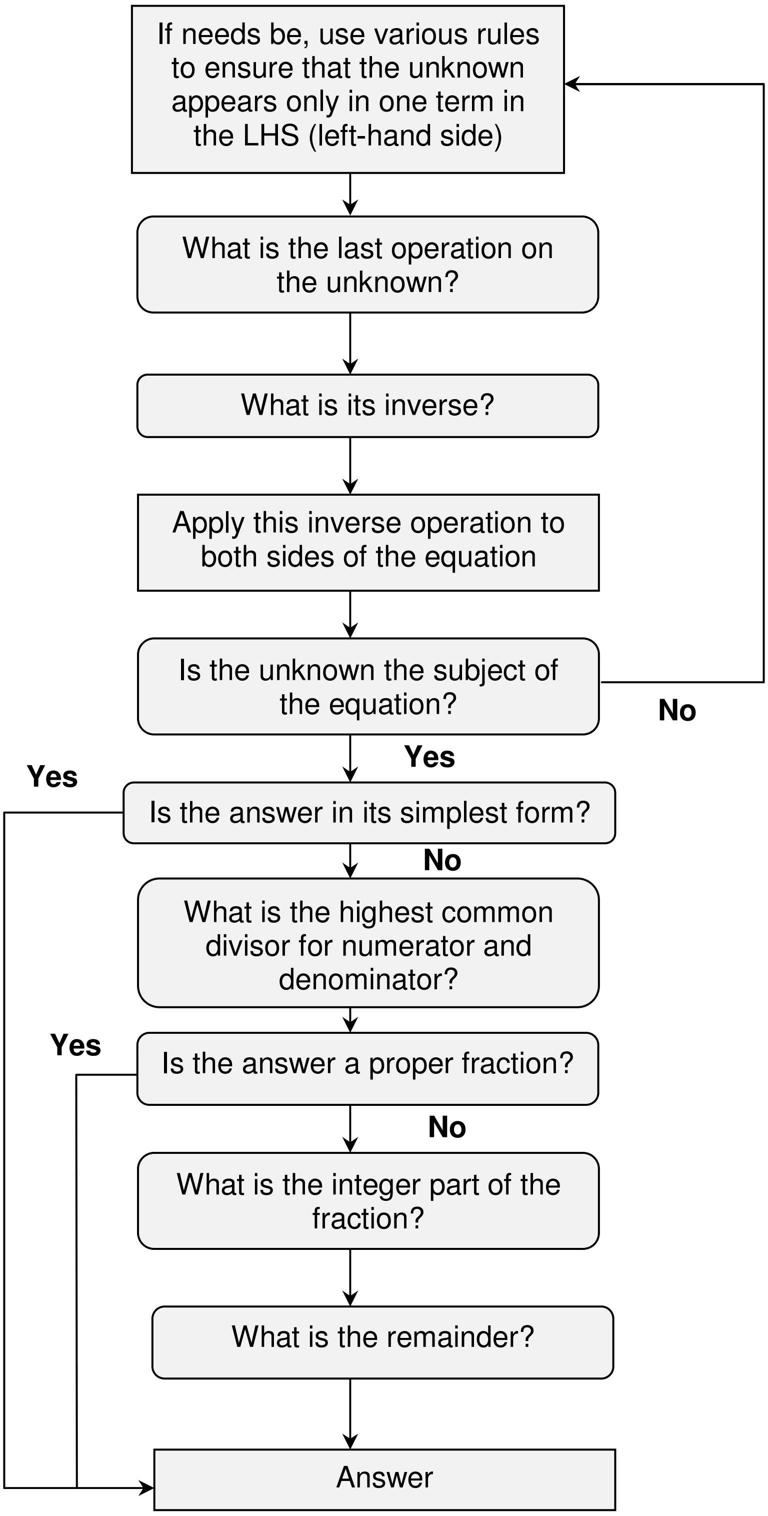


magnitude, $|z|$ **argument,** $\arg(z)$ (on the Argand diagram)

$z = r\ e^{j\theta}$, where $r \geq 0$, θ - angle
(in degrees or radians)

polar radius r polar angle θ (on the ordinary
plane)
(distance to the origin)

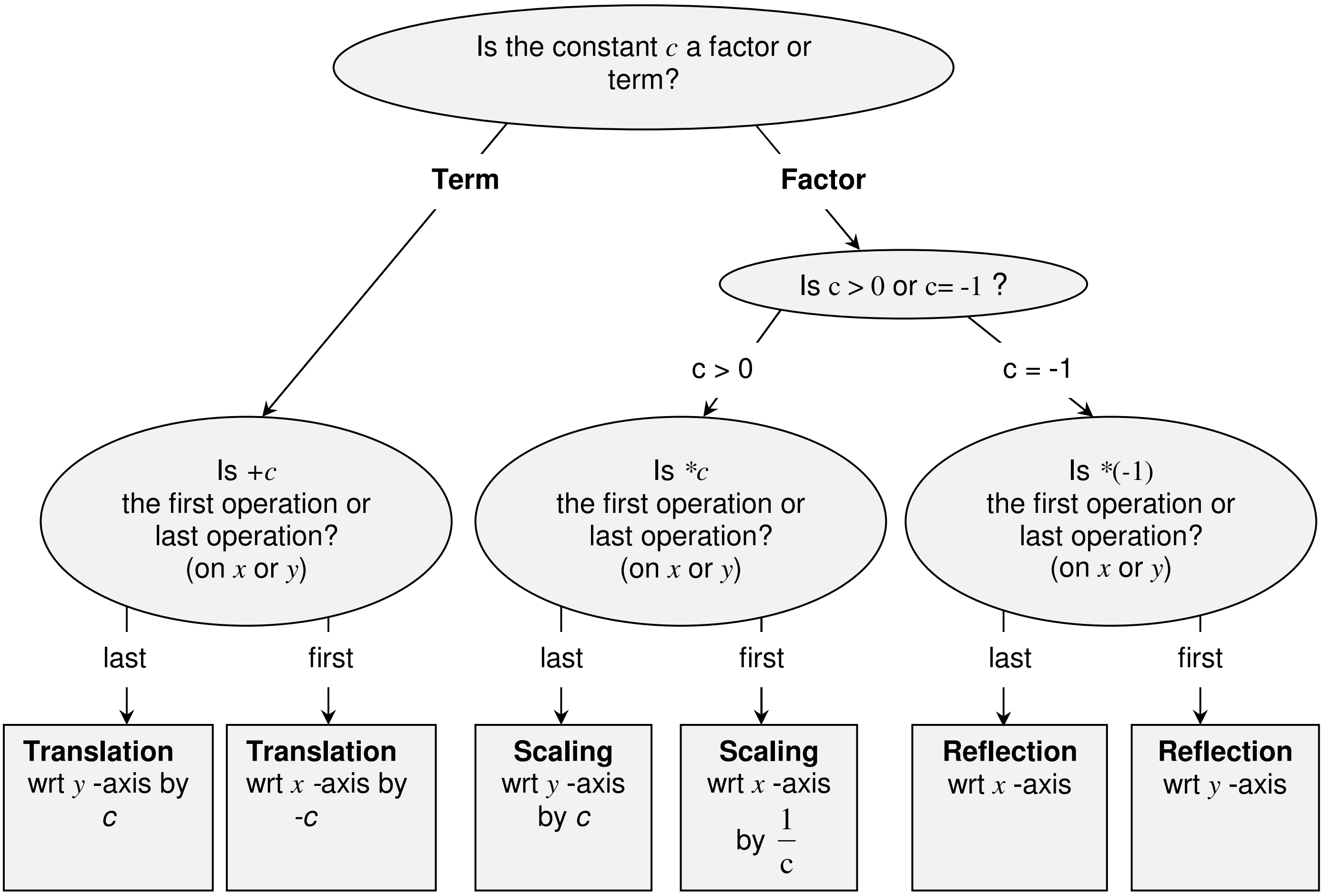
Decision Tree For Solving Simple Equations



Sketching Graphs by Simple Transformations

1. Drop all constant factors and terms.
2. Bring the constants back one by one in the Order of Operations (not necessary but advisable) and at each Step use the Decision Tree given below to decide which simple transformation is affected by each constant. Sketch the resulting graphs underneath one another.

DECISION TREE



Note 1: if c is a **negative factor**, write $c = (-1)*|c|$, so that c affects two simple transformations and not one.
Note 2: if c is affects **neither first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations.
Note 3: if **the same operation** is applied to x in all positions in the equation this operation still should be treated as the first operation, as in $\ln(x - 2)+\sin(x - 2)$.
Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is effected by adding a constant c to y or multiplying it by a constant c, y should be made the subject of the functional equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\frac{1}{c}$, respectively.

IV. GLOSSARY

ABSTRACTION - a general concept formed by extracting common features from specific examples.

ACUTE ANGLE – a positive angle which is smaller than 90^0.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by the highest common DIVISOR.

CIRCLE – a LOCUS of points inside and including a CIRCUMFERENCE.

CIRCUMFERENCE – a LOCUS of points at the same distance from a specific point (called a centre).

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

COMPOSITION – a combination of two or more functions (operations) forming a single new function by applying one to the output of another, e.g. composition of two function '$f\,(\,)$' and '$g\,(\,)$' can be represented as $f\,(g\,(\,))$.

CONCEPT - a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN, e.g. in the EXPRESSION x + a, x is usually understood to represent a VARIABLE and a, a constant (which is independent of this VARIABLE).

If there is only one algebraic CONSTANT, the preferred choice for its algebraic symbol is a. The second choice is b, and the third, c. If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM - a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s,* kg/m^3, £.

DOMAIN – a set of all allowed values of the ARGUMENT.

EQUATION – a mathematical statement involving VARIABLES and the = sign which can be true or false, depending on the values taken by the VARIABLES (which are called UNKNOWNS in this case), e.g. $2x + 3y = 10$ is true when $x = y = 2$ but not when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. $2(a + b)$, $2ab$.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, *a* and *b*.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

FUNCTION – a mathematical OPERATION or a composition of OPERATIONS which establishes a relationship between values of the ARGUMENTS (INDEPENDENT VARIABLES, INPUTS) in its DOMAIN and VALUE OF THE FUNCTION (DEPENDENT variable, output). a function is completely defined only when its domain is defined, e.g. $f(x) = 2x + 3$, x - real' describes a FUNCTION whose SYMBOL is 'f' or '$f(\ )$' (no multiplication is implied) and whose ARGUMENT is called 'x'. Its DOMAIN is 'all reals'. Given any value of 'x', you can find the corresponding value of this FUNCTION by first multiplying the given value of 'x' by 2 and then by adding 3 to the result.

FUNCTION SYMBOL – usually a Latin letter, usually from the first part of the alphabet. If there is only one FUNCTION, the preferred choice for its symbol is 'f'. The second choice is 'g', and the third, 'h'. If there are more variables, then one chooses, in order of preference, 'u', 'v' and 'w'. If more FUNCTIONS are involved, we often make use of subscripts and superscripts, upper case and Greek letters.

GENERALISATION - an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH - a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INDEPENDENT VARIABLE – ARGUMENT, INPUT, e.g. given '$y = f(x)$', 'x' is an INDEPENDENT VARIABLE.

INPUT – (value of) ARGUMENT, (value of) INDEPENDENT VARIABLE, e.g. given '$y = f(x)$', 'x' is INPUT; also given '$y = f(2)$', '2' is INPUT.

INTEGER PART – when dividing a positive integer m into a positive integer n, k is the INTEGER PART if it is the largest positive integer producing $k*m \le n$. The REMAINDER is the difference $n - k*m$, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that (if it exists) undoes what the original OPERATION does

LAST OPERATION – see ORDER OF .OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION – an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. '$2x - 3 = 0$' is a LINEAR EQUATION, '2' is a COEFFICIENT in front of the UNKNOWN and '-3' is a FREE TERM.

LOCUS – a set of all points on a plane (or in space) with a specific property.

NECESSARY - 'A' is a NECESSARY condition of 'B' IF 'B ⇒ A' (B implies A), so that 'B' cannot take place unless 'A' is satisfied.

NON-DIMENSIONAL QUANTITY - a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. '$2 \ln (x) - 3 = 0$' is a NON-LINEAR EQUATION.

OPERATION (mathematical) – something that can be done to CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which the OPERATIONS must be performed. By convention, the ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[( )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for are fractions and functions. Indeed, when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent, and some authors do not bracket ARGUMENTS of elementary FUNCTIONS, such as exp, log, sin, cos, tan *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.* Other OPERATIONS must be performed in the order of decreasing complexity, which is

FUNCTIONS $f(\,)$

POWERS (including inverse operations of roots and logs)

MULTIPLICATION (including inverse operation of division)

ADDITION (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

OUTPUT – (value of) DEPENDENT VARIABLE, (value of the) FUNCITON, e.g. given '$y = f(x)$', 'y' is OUTPUT; also given '$f(x) = 2x + 3$' and '$x = 2$', '7' is OUTPUT (indeed, $2*2+3 = 7$).

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, x, e.g. 'ab' is a PRODUCT, and so is '$(a + b)c$'.

QUOTIENT - a mathematical expression where the last operation is division.

RANGE – the set of all possible values of the function. **HERE**

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the '=' sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SEQUENCE – a function with an INTEGER ARGUMENT.

SIMPLE EQUATION – an EQUATION OF ONE UNKNOWN, LINEAR or NON-LINEAR but such that can be reduced to LINEAR by a simple CHANGE OF VARIABLE.

SIMPLE TRANSFORMATIONS - translation, scaling or reflection - are affected by adding a constant or multiplying by a constant.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT - 'A' is a SUFFICIENT condition of 'B' IF 'A $\Rightarrow$ B' (A implies B), so that if 'A' is satisfied, then 'B' takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is addition, +, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its algebraic symbol is x. The second choice is y and the third, z. If there are more variables, then, one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

V. STUDY SKILLS FOR MATHS

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours** each week:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.

2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**

3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).

4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

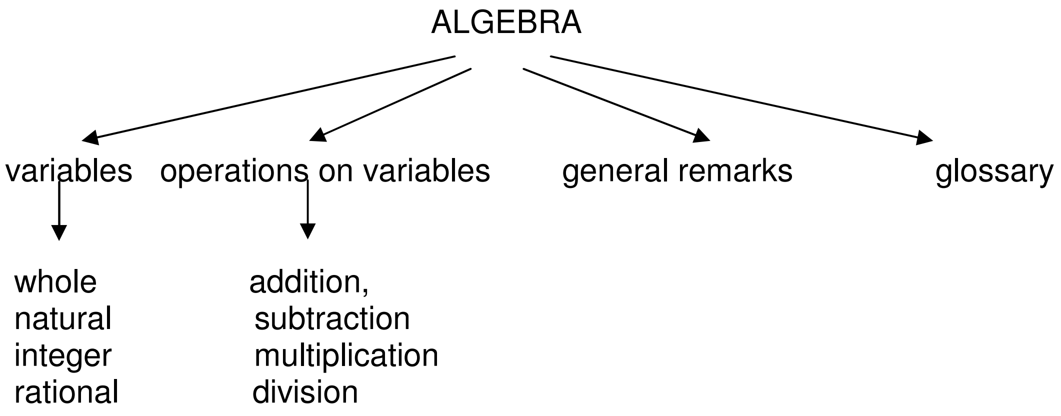
Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

Thus, you can construct the flow chart which looks like



2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.

3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing.** Try this a few times before checking with the notes which step is a problem.

5. Repeat the process for each problem.

6. Repeat the process for each subtopic.

7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.

2. Keep consulting the **Glossary**.

3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.

4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to “make invisible brackets visible”.

2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.

3. The concept of **inverse operation**.

4. The **Decision Tree for Solving Simple Equations**.

5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.

6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real funciton of real variable (real argument). You should know the precise meaning of the word constant (you should always say "constant with respect to (the independent variable) x or t or whatever..."

7. How to do function composition and decomposition using **Order of Operations**

8. How to use graphs

9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos

10. The **Trigonometry Summary**

11. The approximate values of e (≈ 2.71) and π (≈ 3.14)

12. What is j ($=\sqrt{-1}$) and what is j^2 (=-1)

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand digram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

VI. TEACHING METHODOLOGY (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question - Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft's book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGING AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND SOMETHING IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do exercises yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) - rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.
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Q: I also feel that the pace has been too high; we spent only about an hour on sequences and 2 weeks on integration.

A: We have little time to cover the necessary material, and it is spread out accordingly. However, you do not need to know more about sequences than was already discussed.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education - development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you - in its essence - for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise just a sequence of names, facts or dates there are well established techniques promoted in various books on memory (they suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with this objects. However, this technique will not work with technical information. What you need to establish are much deeper - meaningful - connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now - in their full glory - LATER, in year 2 and 3, not to-day.

Hope this helps!