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Basic Calculus Unveiled

18 September 2012

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I. INTRODUCTION

These notes are based on the lectures delivered by the author to engineering students of London South Bank University over the period of 16 years. This is a University of widening participation, with students coming from many different countries, many of them not native English speakers. Most students have limited mathematical background and limited time both to revise the basics and to study new material. A system has been developed to assure efficient learning even under these challenging restrictions. The emphasis is on systematic presentation and explanation of basic abstract concepts. The technical jargon is reduced to the bare minimum.

Nothing gives a teacher a greater satisfaction than seeing a spark of understanding in the students' eyes and genuine pride and pleasure that follows such understanding. The author's belief that most people are capable of succeeding in - and therefore enjoying - the kind of mathematics that is taught at Universities has been confirmed many times by these subjective signs of success as well as genuine improvement in students' exam pass rates. Interestingly, no correlation had ever been found at the Department where the author worked between the students' qualification on entry and graduation.

The book owns a lot to the authors' students – too numerous to be named here - who talked to her at length about their difficulties and successes, e.g. see Appendix VII on Teaching Methodology. One former student has to be mentioned though – Richard Lunt – who put a lot of effort into making this book much more attractive than it would have been otherwise.

The author can be contacted through her website www.soundmathematics.com. All comments are welcome and teachers can obtain there the copy of notes with answers to questions suggested in the text as well as detailed Solutions to suggested Exercises. The teachers can then discuss those with students at the time of their convenience.

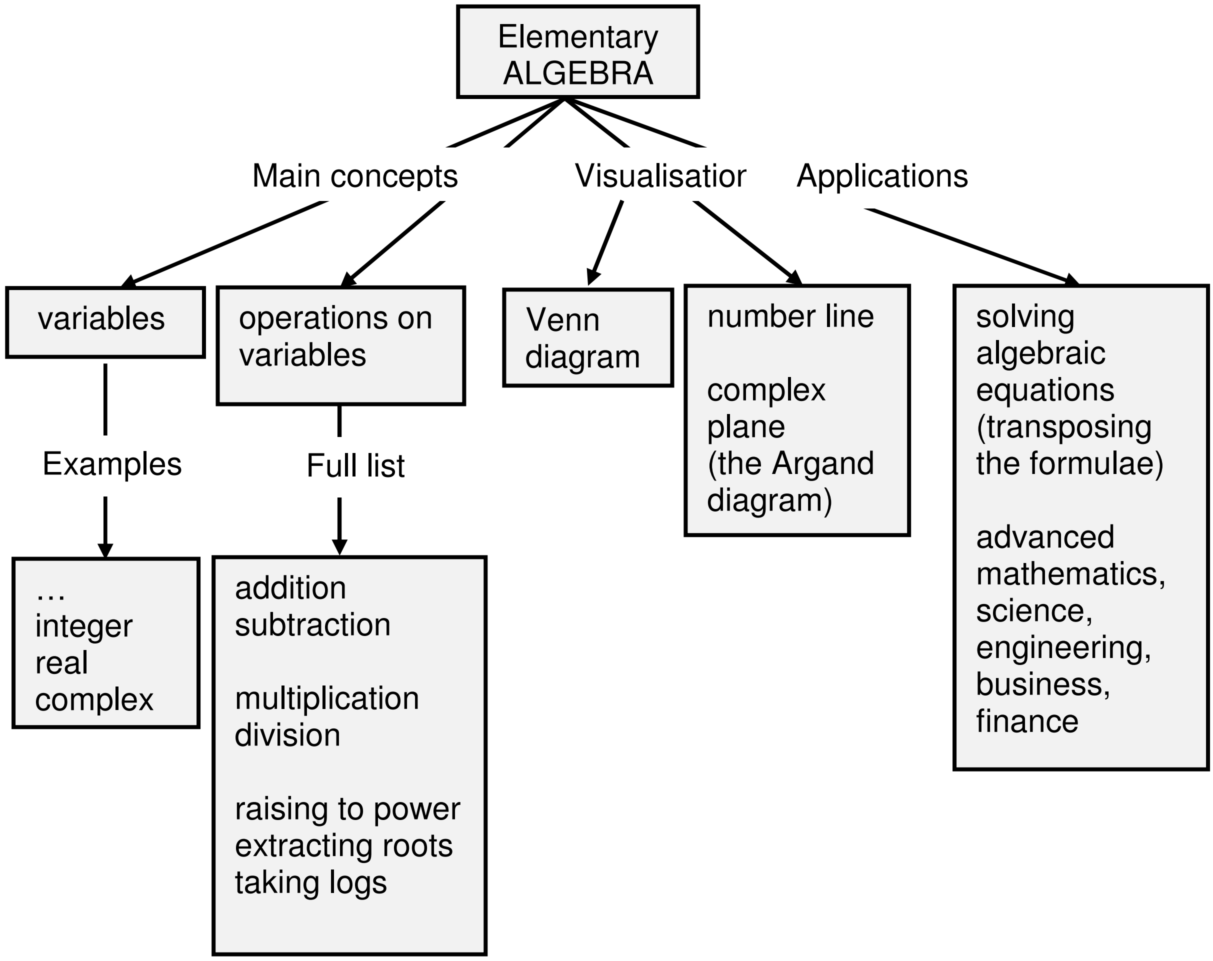
Good luck everyone!

II. CONCEPT MAPS

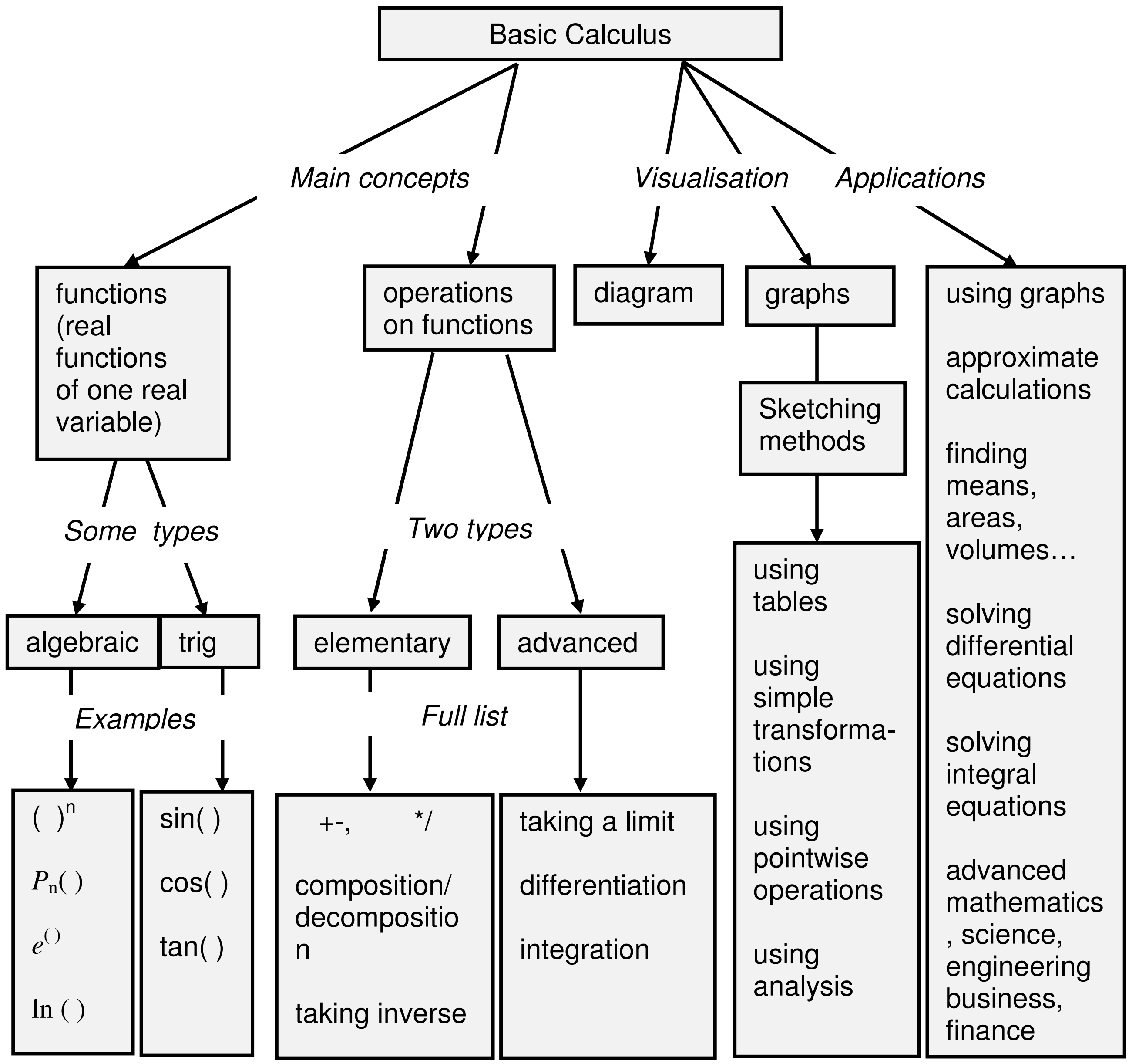
Throughout when we first introduce a new **concept** (a technical word or phrase) or make a conceptual point we use the bold red font. We use the bold blue to verbalise or emphasise an important idea.

Two major topics are covered in these Notes, Differentiation Calculus and Integral Calculus. You can understand this topic best if you first study the Notes on Elementary Algebra and Functions.

Here is a **concept map** of Elementary Algebra.



Below is a concept map of Basic Calculus. It is best to study it before studying any of the Calculus Lectures to understand where it is on the map. The more you see of the big picture the better you learn!



III. LECTURES

Lecture 13. CALCULUS: Sequences, Limits and Series

In Calculus we study **functions and operations on functions** (see Calculus Concept Map). Elementary operations on functions (algebraic operations and composition/decomposition) were covered in Lectures 4 - 7. We now introduce the first **advanced operation on functions** – **taking a limit**. We introduce it first for discrete functions called sequences.

13.1 Sequences

A **sequence** is an ordered set of numbers x_n, a **discrete function** $x(n)$, where the argument is an integer n. This means that a discrete function is a function whose domain is the set of all integers or else its subset. Such argument is called the **counter**.

There are different ways of writing up a sequence: $\{x_n\}$ or $x_1, x_2, x_3, ..., x_N$ (each symbol n, 1, 2, 3, …, N can also be called an **index** or subscript). A sequence can be **finite** (having a fixed number of elements) or **infinite** (whatever index you specify there is always an element in the sequence with a greater index).

13.1.1. Defining a Sequence as Function of Counter

A sequence can be defined (described) by specifying a relationship between each element and its counter.

Examples:

1. Describe sequence $1, 2, 3, 4, ...$ using the above notations.
Solution
$x_1 = 1, x_2 = 2, x_3 = 3, ...$
Question: What is the relationship between the counter and the sequence element?
Answer:
$\Rightarrow \quad x_n = n$

2. Describe sequence $1, 4, 9, 16, ...$ using the above notations.
Solution
$x_1 = 1, x_2 = 4, x_3 = 9, ...$
Question: What is the relationship between the counter and the sequence element?
Answer:
$\Rightarrow x_n = n^2$

13.1.2. Visualising a Sequence as a Function of Counter

If a sequence is defined (described) by specifying a relationship between each element and its counter it can be visualised as a sequence of points on the number line (see figure 13.1) or else as a discrete graph (see figure 13.2).

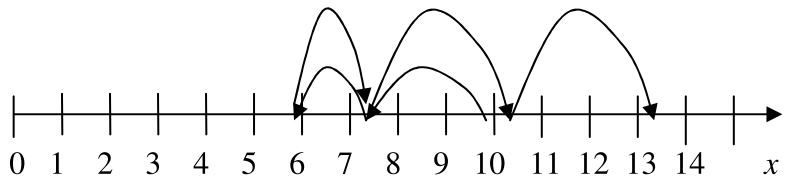


Figure 13.1. An example of a sequence x_n, $n = 0, 1, \ldots, 5$ visualised using the number line. The arrowed arcs are used to indicate the order of elements in the sequence.

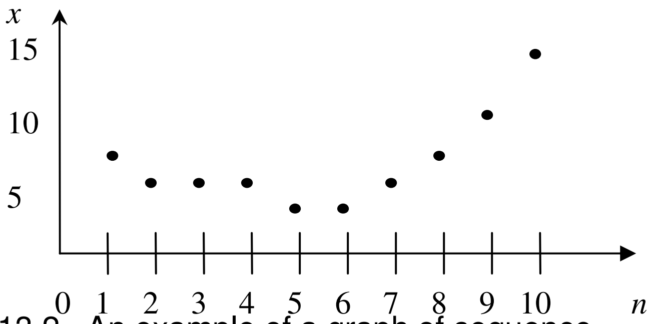


Figure 13.2. An example of a graph of sequence x_n, $n = 0, 1, \ldots, 10$.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

13.1.3 Applications of Sequences

A digital signal x (see figure 13.3 below) has the following characteristics:

1) it holds a fixed value for a specific length of time

2) it has sharp, abrupt changes

3) a preset number of values is allowed.
http://upload.wikimedia.org/wikipedia/commons/8/88/Sampled.signal.svg

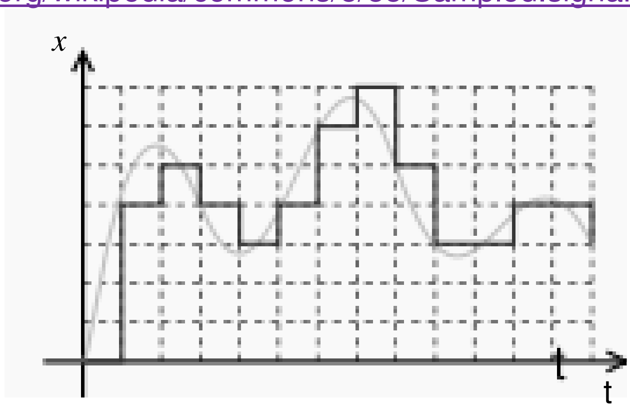


Figure 13.3. A typical digital signal. **http://www.privateline.com/manual/threeA.htm**
Therefore, digital signals are examples of sequences.

13.1.4. Defining a Sequence via a Recurrence Relationship

A sequence element may be defined in another way, via a **recurrence relation**

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_1),$$

where f may be a function of several arguments and not just one argument.

Thus, there are two ways to describe a sequence,

1. using a functional relation $x(n)$, which specifies how each sequence element is defined by its counter;

2. using a recurrence relation $x_{n+1} = f(x_n, x_{n-1}, \ldots, x_1)$, which specifies how each sequence element is defined by previous sequence element(s).

Examples:

1. Given a sequence $x_n = n^2$ we can change the functional description to a recurrence relation

   $$x_{n+1} = (n+1)^2 = n^2 + 2n + 1 \quad \Rightarrow \quad x_{n+1} = x_n + 2\sqrt{x_n} + 1$$

   When given such a recurrence the first element needs to be specified. Only then can we start evaluating other elements.

2. A **Fibonacci sequence**: 1, 1, 2, 3, 5, 8, 13, 21, ... can be described via a recurrence relation

   $$x_{n+1} = x_n + x_{n-1}.$$

   When given such a recurrence the first two elements have to be specified. Only then can we start evaluating other elements. Let us check that the above recurrence describes the given sequence:

Question: What are x_1 and x_2?
Answer:

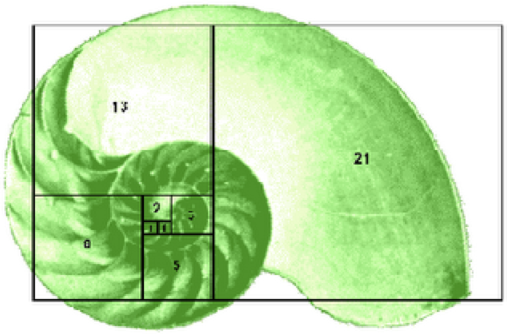
Question: Does x_3 satisfy the given recurrence relation and why?
Answer:

Question: Does x_4 satisfy the given recurrence relation and why?
Answer:

Question: Does x_5 satisfy the given recurrence relation and why?
Answer:

Fibonacci sequences in nature

When superimposed over the image of a nautilus shell we can see a Fibonacci sequence in nature:



http://munmathinnature.blogspot.com/2007/03/fibonacci-numbers.html

Each of the small spirals of broccoli below follows the Fibonacci's sequence.

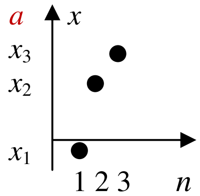
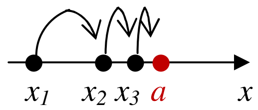
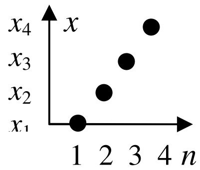
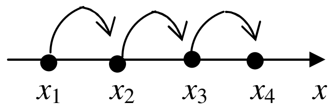
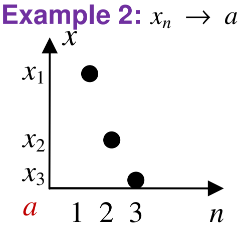
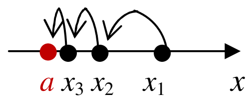
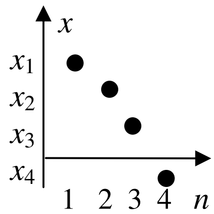
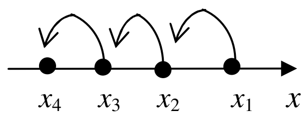
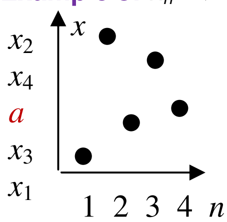
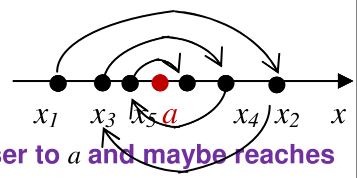
http://www.pdphoto.org/PictureDetail.php?mat=pdef&pg=8232

13.2 Limit of a sequence

Taking a limit of a sequence as n grows larger and larger without bounds is the first **advanced operation on function** that we cover. In mathematics, instead of the phrase n *grows larger and larger without bounds* we use the shorthand $n \to \infty$ (verbalised as n tends to infinity). If it exists the outcome of applying this operation to a (discrete) function x_n is called $\lim x_n$ and is either a number or else $\pm\infty$ (either +**infinity** or -**infinity**). Sometimes instead of $\lim x_n$ we write $\lim\limits_{n\to\infty} x_n$. However, the description $n \to \infty$ is usually understood and not mentioned.

Note: ∞ is not a number but a symbol of a specific sequence behavior, $+\infty$ means that the sequence increases without bounds and $-\infty$ means that the sequence decreases without bounds - see the right column in the Table in Section 13.2.1 below.

13.2.1 Definition of a Limit of a Sequence

Examples of a finite limit $\lim\limits_{(n\to\infty)} x_n = a$ (can also write $x_n \underset{(n\to\infty)}{\to} a$)	Examples of an infinite limit $\lim\limits_{(n\to\infty)} x_n = \pm\infty$ (can also write $x_n \underset{(n\to\infty)}{\to} \pm\infty$)
Example 1: $x_n \to a-$  or  x_n gets closer and closer to a from below and maybe reaches it.	**Example 1:** $x_n \to +\infty$  or  For any index N we can find a greater index n such that $x_n > x_N$ (sequence increases without bounds)
Example 2: $x_n \to a+$  or  x_n gets closer and closer to a from above and maybe reaches it.	**Example 2:** $x_n \to -\infty$  or  For any index N we can find a greater index n such that $x_n < x_N$ (sequence decreases without bounds)
Example 3: $x_n \to$ a  or  x_n gets closer and closer to a and maybe reaches it.	

Examples: Sketch the sequence to find its limit:

1. $x_n = \dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n come closer and closer to 0.
2. $x_n = -\dfrac{1}{n} \to 0$ **Verbalise:** as n increases, elements x_n come closer and closer to 0.
3. $x_n = \dfrac{(-1)^{n+1}}{n} \to 0$ **Verbalise:** as n increases, x_n come closer and closer to 0.
4. $x_n = n \to \infty$ **Verbalise:** as n increases, sequence elements increase without bounds.
5. $x_n = -n \to -\infty$ **Verbalise:** as n increases, sequence elements decrease without bounds.

Sometimes, limits can be found using analytical considerations and sometimes, using the **Table**,
Rules and a **Decision Tree for Limits**:

13.2.2 Table of Limits

x_n	$\lim x_n$
$1/n$	0
$a^{1/n}$, $a > 0$	1
q^n, $\lvert q \rvert < 1$	0
$(1+\frac{1}{n})^n$	e

easy to find

the proof is rather involved

If $\{x_n\}$ has a finite limit the sequence is said to **converge**.

13.2.3 Rules for (Properties of) Limits

1. The limit is uniqie.

2. $\lim(\alpha x_n + \beta y_n) = \alpha \lim x_n + \beta \lim y_n$, α, β – constant
Verbalise: limit of a sum is a sum of limits and any constant factor can be taken out – **linearity property.**

3. $\lim x_n y_n = \lim x_n \lim y_n$
Verbalise: limit of a product is a product of limits.

4. $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$, $\lim y_n \neq 0$
Verbalise: limit of a ratio is a ratio of limits.

Laws 2 - 4 give a general recipe for evaluating a limit of a composition of sequences **– instead of the sequence element substitute its limit.** That is, the Laws 2 – 4 state that if a limit exists the operation of taking the limit and any algebraic operation can be performed in an arbitrary order.

Optional

5. $x_n \geq y_n \Rightarrow \lim x_n \geq \lim y_n$ - **every element of one sequence is greater than or equal to the element with the same index in the other sequence** $\Rightarrow$ **its limit is also greater than or equal to the limit of the other sequence.**

6. $cx_n \leq y_n \leq z_n \Rightarrow \lim x_n = \lim z_n \Rightarrow \lim y_n = \lim x_n$
a sequence squeezed between two others with the same limit has the same limit.

Example: $x_n = (-1)^{n+1} n$. $\lim x_n$?

Solution: $\lim x_n$ does not exist $\Leftarrow$ Rule 1

13.2.4 Indeterminancy

Sometimes using the above rules leads to an **indeterminacy** (no obvious answer) like

$$\frac{0}{0}, \frac{\infty}{\infty}, \; 0\cdot\infty, \; \infty-\infty, \; 0^0, \; 1^{\infty} \; \textit{etc.}$$ - **MEMORISE**

Question: Why do these outcomes present no clear answer?
Answer:

An indeterminacy can often be **resolved**, i.e. a clear answer can be found by using tricks.

Examples:

$\frac{0}{0}\downarrow$ **use algebraic trick – FLIP RULE**

1. $x_n = \frac{1}{n^2}, \; y_n = \frac{1}{n} \Rightarrow \lim \frac{x_n}{y_n} = \lim \frac{\frac{1}{n^2}}{\frac{1}{n}} = \lim \frac{1}{n} = 0$

$\frac{0}{0}\downarrow$ **use algebraic trick – FLIP RULE**

2. $x_n = \frac{1}{n^2}, \; y_n = \frac{1}{n} \Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{\frac{1}{n}}{\frac{1}{n^2}} = \lim n = \infty$

3. $x_n = n, \; y_n = n^2$

$\infty-\infty\downarrow$ **use algebraic trick - factorisation**

$\Rightarrow \lim(x_n - y_n) = \lim(n - n^2) = \lim n(1-n) = \lim n \lim(1-n) = \infty\cdot(-\infty) = -\infty$

4. $x_n = n^2, \; y_n = n^2 - 1$

$\frac{\infty}{\infty}\downarrow$ **use algebraic trick – divide top and bottom by the highest power**

5. $\Rightarrow \lim \frac{y_n}{x_n} = \lim \frac{n^2-1}{n^2} = \lim \frac{n^2 - \cancel{1} \div n^2}{n^2} = \lim \frac{1-\frac{1}{n^2}}{1} = 1 - 0 = 1$

13.2.5 Decision Tree for Limits

Decision Tree (a Flow Chart) that can be helpful when evaluating limits is given in figure 13.1. Start reading it from the very top and follow the arrows as dictated by your answers. In many situations the tricks necessary to find a limit are very involved. However, in standard undergraduate courses they require only the use of simple algebraic and trigonometric laws and formulae

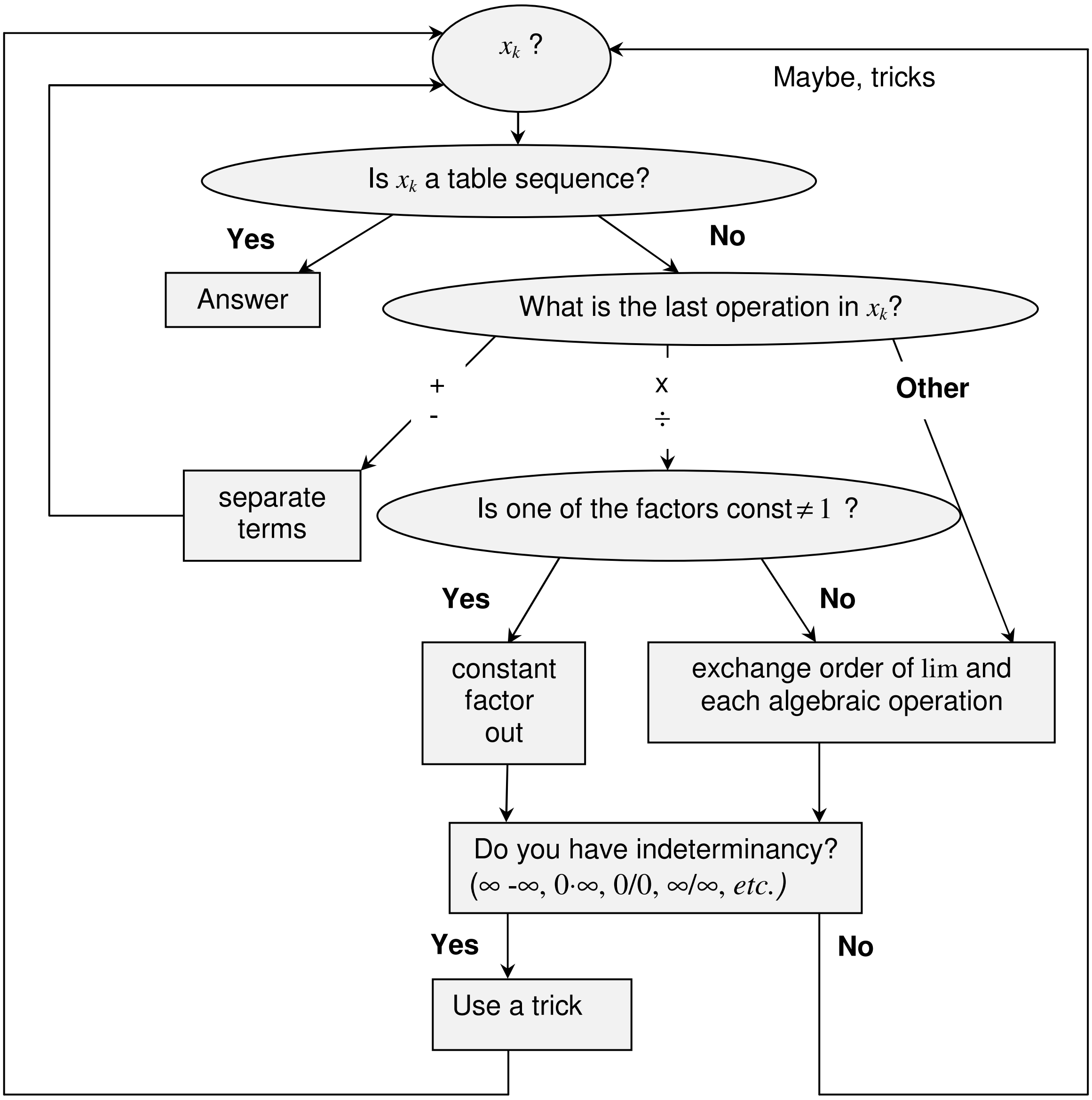


Figure 12.1. Decision Tree for Limits.

13.3 Series

A **series** is a sum of ordered terms (a sum of sequence elements),
Question: What is a sum?
Answer:

A series $x_1 + ... + x_N$ can be written using the shorthand S_N or $\sum_{n=1}^{N} x_n$. In the latter case we say that we use the **sigma notation**, because $\sum$ is a capital Greek letter called *sigma*.

13.3.1 Arithmetic Progression

An arithmetic progression is a sequence defined by the following recurrence relationship: $x_{n+1} = x_n + a$, where a is constant (with respect to n). a is called the **common difference.**

$\Rightarrow S_N = x_1 + ... + x_N = N\dfrac{x_1 + x_N}{2}$. **MEMORISE**

13.3.2 Geometric Progression

A geometric progression is a sequence defined by the following recurrence relationship: $x_{n+1} = ax_n$, where where a is constant (with respect to n). a is called the **common ratio.**

$$\Rightarrow S_N = \frac{x_1(1-a^N)}{1-a}$$

If $|a| < 1$, then $S_N = \dfrac{x_1}{1-a} - \dfrac{x_1 a^N}{1-a} \to \dfrac{x_1}{1-a} \Rightarrow \lim S_N = \dfrac{x_1}{1-a}$. **MEMORISE**

13.4 Instructions for self-study

- **Revise Summaries on ALGEBRA, FUNCTIONS and TRIGONOMETRY**
- **Revise Lecture 11 and study Solutions to Exercises in Lecture 11 using the STUDY SKILLS Appendix**
- **Revise Lecture 12 using the STUDY SKILLS Appendix**
- **Study Lecture 13 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find the following limits:

a) $\lim \dfrac{n^2+n+1}{2n+1}$;

b) $\lim \dfrac{n^2+n+1}{2n^2+1}$;

c) $\lim \dfrac{n^2+n+1}{2n^3+1}$

Q2. Find the sum of a hundred terms of the arithmetic series with first term 1 and common difference 2.

Q3. A steel ball-bearing drops on to a smooth hard surface from a height h. The time to first impact is $T = \sqrt{\dfrac{2h}{g}}$, where g is the acceleration due to gravity. The times between successive bounces are $2eT, 2e^2T, 2e^3T,...,$ where e is the coefficient of restitution between the ball and the surface $(0<e<1)$. Find the total time taken up to the fifth bounce if $T = 1$ and $e = 0.1$.

Q4 (**advanced**). The irrational number π can be defined as the area of a unit circle (circle of radius 1). Thus, it can be defined as a limit of a sequence of known numbers a_n. The method used by Archimedes was to inscribe in the unit circle a sequence of regular polygons (that is, polygons with equal sides – see figure 8.1). As the number of sides increases the polygon "fills" the circle. For any given number n find the area a_n of the regular polygon inscribed in the unit circle. Show, by use of trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras theorem), that the area a_n satisfies the recurrence relation

$$2\left(\frac{a_{2n}}{n}\right)^2 = 1 - \sqrt{\left[1 - \left(\frac{2a_n}{n}\right)^2\right]}, \; n \geq 4.$$

Use trigonometry to show that $a_4 = 2$ and use the above recurrence relation to find a_8.

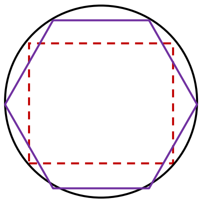


Figure 8.1. A circle with two inscribed regular polygons, one with four sides (a square) and one, with six (a hexagon).

Q5. Explain why $\sum_{k=1}^{\infty} x[k] = \sum_{n=1}^{\infty} x[n] = \sum_{k=0}^{\infty} x[k+1]$.

Lecture 14. DIFFERENTIAL CALCULUS: Limits, Continuity and Differentiation of Real Functions of One Real Variable

14.1 Limits

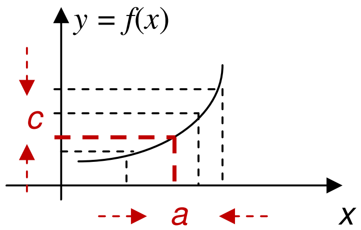
Taking a limit is **an advanced operation on functions**. The outcome of this operation is a number (more generally, a constant with respect to the control variable). The operation involves finding a limiting value of the dependent variable when the independent variable approaches (tends to) a specified limiting value of its own or else to ∞ or **-** ∞.

14.1.1 Definition of a Limit of a Real Function of One Real Variable

Four limiting behaviours are possible:

1. $\lim_{x \to a} f(x) = c$

Verbalise: as x tends to a finite limit a, $f(x)$ tends to a finite limit c.



2. $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$

Verbalise: as x tends (goes off) to infinity, f(x) tends to a finite limit c.

3. $\lim_{x \to \infty} f(x) = \infty$ or $-\infty$, $\lim_{x \to -\infty} f(x) = \infty$ or $-\infty$,

Verbalise: as x tends (goes off) to + or - infinity, $f(x)$ tends to + or - infinity.

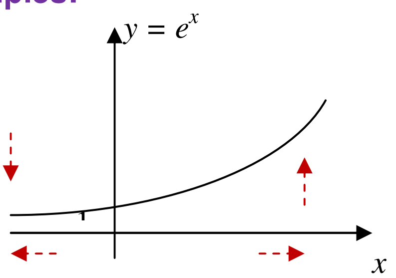
4. $\lim_{x \to a} f(x) = \infty$ or $-\infty$

Verbalise: as x tends to (approaches) a finite limit a, $f(x)$ tends to + or - infinity.

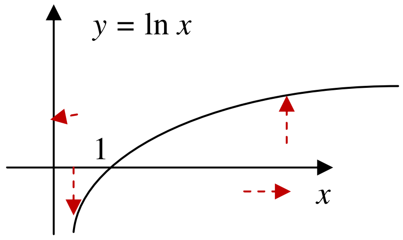
Note: symbols $\pm\infty$ are not numbers, they describe a specific function behaviour.

Examples:

1.



$\lim_{x \to -\infty} e^x = 0$ $\qquad$ $\lim_{x \to \infty} e^x = \infty$

2. 

$\lim_{x\to 0+} \ln x = -\infty$ $\qquad\qquad$ $\lim_{x\to\infty} \ln x = \infty$

14.1.2 Table of Limits of Functions

$f(x)$	$\lim_{x\to 0} f(x)$
$\dfrac{\sin x}{x}$	1
$\dfrac{e^x - 1}{x}$	1

the proof is rather involved

14.1.3 Rules for (Properties of) Limits

1. The limit is unique (that is, there can be only one limiting value).
2. $\lim_{x\to a}[f_1 \begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix} f_2(x)] = \lim_{x\to a} f_1(x) \begin{smallmatrix}+\\-\\\times\\\div\end{smallmatrix} \lim_{x\to a} f_2(x)$ if all limits exist

Note: division is allowed only if $\lim_{x\to a} f_2(x) \neq 0$

Using these rules is equivalent to just substituting a for x.

Optional

3. $f_1(x) \le f_2(x) \le f_3(x)$ and $\lim_{x\to a} f_1(x) = \lim_{x\to a} f_3(x) \;\Rightarrow\; \lim_{x\to a} f_2(x) = \lim_{x\to a} f_1(x)$

Examples:

1. $\lim_{x\to 1}(x^2 - 4x + 8) = 1^2 - 4\cdot 1 + 8 = 5$ **(instead of x we substituted its limiting value)**
2. $\lim_{x\to \pi/2} \sin x = 1$ **(instead of x we substituted its limiting value)**
3. $\lim_{x\to \pi/2} \cos x = 0$ **(instead of x we substituted its limiting value)**

4. $\lim\limits_{x\to\pi/2} \tan x$ does not exist, but $\lim\limits_{x\to\pi/2^{\pm}} \dfrac{\sin x}{\cos x} = \mp\infty$, which is a shorthand for two statements, $\lim\limits_{x\to\pi/2+} \dfrac{\sin x}{\cos x} = -\infty$ and $\lim\limits_{x\to\pi/2-} \dfrac{\sin x}{\cos x} = +\infty$

5. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = 4$

Verbalise: zero over zero indeterminacy – we resolve it using algebraic tricks.

Sometimes, limits can be found using analytical considerations and sometimes, using the **Table**, **Rules** and a **Decision Tree for Limits** (practically the same as the one for sequences).

14.1.4 Applications of the Concept of a Limit

1. "Imagine a person walking over a landscape represented by the graph of $y = f(x)$. Her horizontal position is measured by the value of x, much like the position given by a map of the land or by a global positioning system. Her altitude is given by the coordinate y. She is walking towards the horizontal position given by $x = a$. As she does so, she notices that her altitude approaches L. If later asked to guess the altitude over $x = a$, she would then answer L, even if she had never actually reached that position."
http://en.wikipedia.org/wiki/Limit_of_a_function#Motivation

2. It is often important for engineers to assess how a measured quantity (current, voltage, density, load) behaves with time: does it grow without bounds (then its limit is ∞), tends to a specific value (then its limit is a finite number), oscillates (then it has no limit) *etc.*

14.1.5 Existence of a Limit of a Function

A limit of a function does not always exist.

Examples:

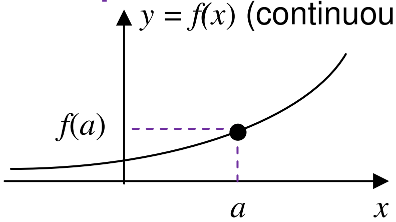
1. $\lim\limits_{x\to\infty} \sin x$ - does not exist.

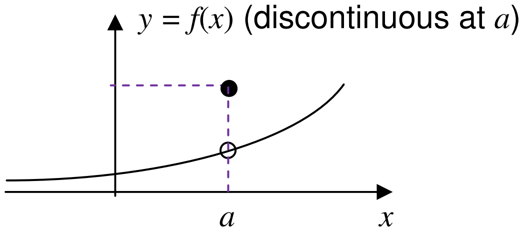
2. $\lim\limits_{x\to\infty} \sin\dfrac{1}{x} = 0.$

14.2 Continuity of a function

A function $f(x)$ is said to be **continuous** at point a if $\lim\limits_{x\to a} f(x) = f(a)$.

Examples:





Note: the filled circle indicates the point that belongs to the graph and an empty circle – the point that does not.

If a function is continuous at each point of an interval it is **continuous on** this **interval**. **Intuitively, a continuous function is a function for which, smaller and smaller changes in the input (argument or independent variable) result in smaller and smaller changes in the output (value of a function or dependent variable).** Otherwise, a function is said to be **discontinuous**.

If $f_1(x)$ and $f_2(x)$ are continuous on interval, so is $f_1(x)$ $+, -, \times, \div$ $f_2(x)$ (provided $f_2(x) \neq 0$ when dividing).

14.2.1 Applications of Continuous Functions

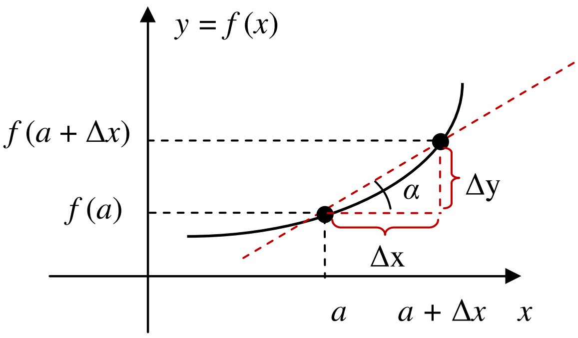
An analogue signal is a continuous signal for which the time varying feature of the signal is a representation of some other time varying quantity... For example, in sound recording, fluctuations in air pressure (that is to say, sound) strike the diaphragm of a microphone which causes corresponding fluctuations in a voltage or the current in an electric circuit. The voltage or the current is said to be an "analog" of the sound.
http://en.wikipedia.org/wiki/Analog_signal

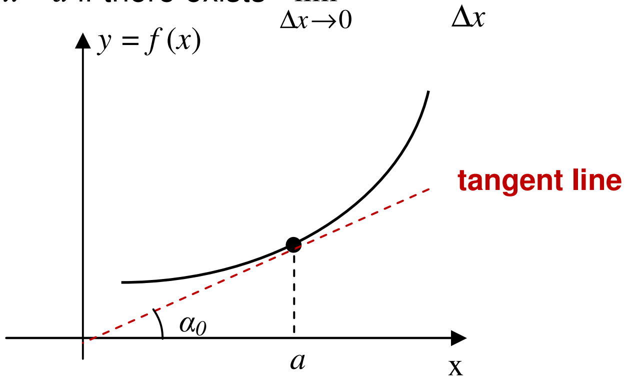
14.3 Differentiation of a real function of one real variable

Differentiation is the second advanced operation on functions we cover. The outcome of this operation is a function, which is called **the derivative** of the original function.

14.3.1 A Derivative of a Function

A continuous function $f(x)$ has a **derivative** at $x = a$ if there exists $\lim_{\Delta x \to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x} \overset{\frac{0}{0}}{=}$





$= \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan\alpha = \tan\alpha_0 \equiv \dfrac{dy}{dx}(a) \equiv \dfrac{df}{dx}(a) \equiv f'(a)$ This limit is called a **derivative**.

Thus, we arrived at the following **definition of the derivative**:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

14.3.2 Geometrical Interpretation of a Derivative

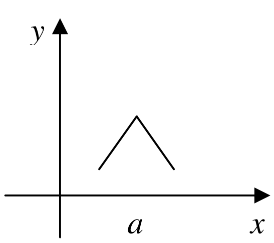
A derivative of a function is a local slope (gradient) of the function at a point. More specifically, it is the slope of the line tangent to the graph of the function at this point.

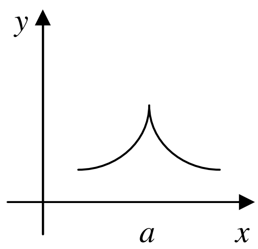
14.3.3 Mechanical Interpretation of a Derivative

A derivative of a function is a local rate (speed) of change of the function at a point.

14.3.4 Existence of a Derivative

A derivative of a function does not always exist.
Here are graphs of functions which have no derivative at point a of their domain:

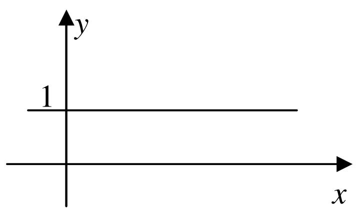




Examples:

1. Differentiate $y = c$ (constant)

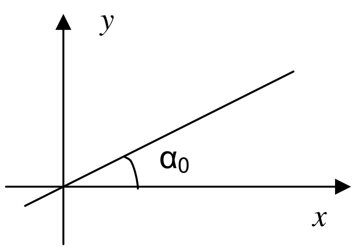
   Can differentiate using a graph: the slope is the same everywhere and is zero.



$$\Rightarrow \frac{dy}{dx} = 0$$

2. $y = c\,x$

   a) Can differentiate using a graph: the slope is the same everywhere – c



$$\Rightarrow \frac{dy}{dx} = c$$

   b) Can differentiate using the basic definition:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c(x+\Delta x) - cx}{\Delta x} = c$$

14.4 A historical note

"When Newton and Leibniz first published their results on calculus in the 17the century, there was great controversy over which mathematician (and therefore which country) deserved credit. Newton derived his results first, but Leibniz published first. Newton claimed Leibniz stole ideas from his unpublished notes, which Newton had shared with a few members of the Royal Society. This controversy divided English-speaking mathematicians from continental mathematicians for many years, to the detriment of English mathematics. A careful examination of the papers of Leibniz and Newton shows that they arrived at their results independently, with Leibniz starting first with integration and Newton with differentiation. Today, both Newton and Leibniz are given credit for developing calculus independently. It is Leibniz, however, who gave the new discipline its name. Newton called his calculus "the science of fluxions". http://en.wikipedia.org/wiki/Calculus

14.5 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 12 and study Solutions to Exercises in Lecture 12 using the STUDY SKILLS Appendix**
- **Revise Lecture 13 using the STUDY SKILLS Appendix**
- **Study Lecture 14 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (using $\lim_{x\to 0}\frac{\sin x}{x}=1$ and trigonometric identities)

b) $\lim_{x\to 0}\frac{e^x - e^{-x}}{2}$

c) $\lim_{x\to 0}\frac{\sin^2 3x}{x^3}$

d) $\lim_{x\to 0}\frac{x^2-1}{2x^2+3}$

e) $\lim_{x\to \infty}\frac{x^2-1}{2x^2+3}$

Q2. Discuss continuity of $y = \tan x$.

Q3. Plot $y = e^x$. Differentiate this function approximately using the graph and plot the approximate derivative $\frac{d}{dx}e^x$.

Q4. Prove

a) $\frac{d}{dx}x^n = nx^{n-1}$ (Hint: use the Binomial Theorem – look it up in books or on google)

b) $\frac{d}{dx}e^x = e^x$

c) $\dfrac{d}{dx}\sin\ x = \cos\ x$

d) $\dfrac{\mathrm{d}}{\mathrm{dx}}\cos \mathrm{x} = -\sin \mathrm{x}$

Lecture 15. DIFFERENTIAL CALCULUS: Differentiation (ctd.)

Using **the first principles** (that is, the definition of a derivative) we can create a **table of derivatives of elementary functions**.

15.1 Differentiation Table

function to differentiate

argument

differentiation variable

$y = f(x)$	$\dfrac{df(x)}{dx}$
const	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

This row means $\dfrac{d\,\mathrm{const}}{dx} = 0$

This row means $\dfrac{d\,\mathrm{x}^{\mathrm{n}}}{dx} = \mathrm{nx}^{\mathrm{n-1}}$

This row means $\dfrac{d\,e^x}{dx} = e^x$

This row means $\dfrac{d\ln \mathrm{x}}{dx} = \dfrac{1}{x}$

This row means $\dfrac{d\sin \mathrm{x}}{dx} = \cos x$

This row means $\dfrac{d\cos \mathrm{x}}{dx} = -\sin x$

Examples:

1. $\dfrac{\mathrm{d}\cos \pi}{\mathrm{dx}} = 0$

> **Differentiation variable** – x
> **Function to differentiate** – constant
> **It is the table function**

2. $\dfrac{\mathrm{d}\,t^3}{\mathrm{d}t} = 3t^2$

> **Differentiation variable** – t
> **Function to differentiate** – power 3
> **It is the table function of** t

3. $\dfrac{d\sin v}{dv} = \cos v$

Differentiation variable – v
Function to differentiate – $\sin(\ )$
It is the table function of v

4. $\dfrac{d(\ln t)^3}{d(\ln t)} = 3(\ln t)^2$

Differentiation variable – $\ln t$
Function to differentiate – power 3
It is the table function of $\ln t$

5. $\dfrac{d\cos e^v}{de^v} = -\sin e^v$

Differentiation variable – e^v
Function to differentiate – $\cos(\ )$
It is the table function of e^v

To differentiate combinations of elementary functions we can use **Differentiation Table, Dfferentiation Rules** and **Decision Tree** given in figure 15.1.

15.2 Differentiation Rules

$$\left.\begin{array}{l} 1.\ \dfrac{d\alpha f(x)}{dx} = \alpha\dfrac{df(x)}{dx} \text{ - constant factor out rule} \\ 2.\ \dfrac{d[f(x)+g(x)]}{dx} = \dfrac{df(x)}{dx} + \dfrac{dg(x)}{dx} \text{ – sum rule} \end{array}\right\} \text{linearity property}$$

3. $\dfrac{d(f(x)g(x))}{dx} = \dfrac{df(x)}{dx}g(x) + \dfrac{dg(x)}{dx}f(x)$ – **product rule**

4. $\dfrac{d\dfrac{f(x)}{g(x)}}{dx} = \dfrac{\dfrac{df(x)}{dx}g(x) - \dfrac{dg(x)}{dx}f(x)}{g^2}$ – **quotient rule**

5. $\dfrac{d(f(g(x))}{dx} = \dfrac{df(g)}{dg}\dfrac{dg}{dx}$ – **chain rule (decompose, differentiate, multiply).** Here $g = g(x)$

Note: These rules can be proven using the definition of the derivative as a limit and rules for limits.

Examples:

1. $$\frac{d\ 2\ \sin\ x}{dx} = 2\cos\ x$$

Differentiation variable – x
Function to differentiate – $2\sin x$
Last operation in this function – multiplication
One of the factors constant
Apply "constant factor out" rule

Constant factor $\alpha = 2$

Variable Factor $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

2. $$\frac{d\ (\ \sin\ x\ + \cos\ x)}{dx} = \cos x - \ \sin x$$

Differentiation variable – x
Function to differentiate – $3\sin x + 5\cos x$
Last operation in this function – addition
Apply the sum rule

---

f-line: 1st term $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

g-line: 2nd term $g(x) = \cos x$, $\dfrac{d\,g(x)}{dx} = \dfrac{d\ \cos\ x}{dx} = -\sin x$

3. $$\frac{d\ (x^2 +\ 3x\ + 2\ )}{dx} = \frac{d\ x^2}{dx} + \frac{d\ 3x}{dx} + \frac{d\ 2}{dx} = 2x + 3$$

Differentiation variable – x

Function to differentiate – $x^2 + 3x + 2$
Last operation in this function – addition
Apply the sum rule
Then go over the same reasoning for each term

4. $\dfrac{d\,(x\ \sin\ x)}{dx} = \sin\ x + x\ \cos\ x$

Last operation in this function – multiplication
Is one of the factors constant? No
Apply the product rule

f-line: 1st factor $f(x) = x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\,x}{dx} = 1$

g-line: 2nd factor $g(x) = \sin x$, $\dfrac{d\,g(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

5. $\dfrac{d\,(x\ \ln\ x)}{dx} = \ln\ x + 1$ (using the product rule)

f-line: $f(x) = x$, $\dfrac{df(x)}{dx} = \dfrac{dx}{dx} = 1$

g-line: $g(x) = \ln x$, $\dfrac{dg(x)}{dx} = \dfrac{d\ln x}{dx} = \dfrac{1}{x}$

6. $\dfrac{d\,(\dfrac{\sin\ x}{x})}{dx} = \dfrac{x\ \cos\ x\ -\ \sin\ x}{x^2}$ (using the quotient rule)

f-line: Numerator $f(x) = \sin x$, $\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin\ x}{dx} = \cos x$

g-line: Denominator $g(x) = x$, $\dfrac{d\,g(x)}{dx} = \dfrac{dx}{dx} = 1$

7. $\dfrac{d\left(\dfrac{\ln x}{x}\right)}{dx} = \dfrac{x \cdot \dfrac{1}{x} - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$

Differentiation variable? x

f(x) ? $\dfrac{\ln x}{x}$

Last operation in this function ? Division

Apply which rule? quotient rule

---

f-line: Numerator $f(x)=\ln x,\ \dfrac{d\,f(x)}{dx} = \dfrac{d\ \ln x}{dx} = \dfrac{1}{x}$

g-line: Denominator $g(x)=x,\ \dfrac{d\ g(x)}{dx} = \dfrac{dx}{dx} = 1$

8. $\dfrac{d\ \sin\ 2x}{dx} = 2\cos 2x$

Differentiation variable? x

f(x)? $\sin 2x$

Last operation in this function? $\sin(\ )$

Its argument? $2\ x$

Apply which rule? The chain rule

---

f-line: Last operation $f()=\sin(\ ),$

$\dfrac{d\,f(x)}{dx} = \dfrac{d\ \sin(g)}{dg} = \cos g = \cos 2x$

g-line: Its argument $g(x)=2x,\ \dfrac{d\ g(x)}{dx} = \dfrac{d\ 2x}{dx} = 2$

9. $\frac{d\ \ln\ 3x}{dx} = 3\frac{1}{3x} = \frac{1}{x}$

Differentiation variable – x
Function to differentiate – $\ln 3x$
Last operation in this function – $\ln(\ )$
Its argument – $3x$
Apply the chain rule

f-line: Last operation $f(\ ) = \ln(\ ),\ \frac{d\,f(x)}{dx} = \frac{d\ \ln(g)}{dg} = \frac{1}{g} = \frac{1}{3x}$

g-line: Its argument $g(x) = 3x,\ \frac{d\,g(x)}{dx} = \frac{d\ 3x}{dx} = 3$

15.3 Decision Tree for Differentiation

Generally, to differentiate a function we use the **Decision Tree for Differentiation** (see figure 15.1 below) that allows us to decide whether to use the Differentiation Table (containing derivatives of elementary functions) straight away or first use Differentiation Rules (on how to differentiate combinations of elementary functions). Once the rule is applied, even if the problem is not solved, it is reduced to one or several simpler problems which have to be treated using the Decision Tree again.

To use the Decision Tree start at the top and follow the arrows which are associated with the correct answers (if any). Use it as a formula, substituting your differentiation variable for x and your function for $f(x)$.

15.4 The higher order derivatives

If the derivative of $f(x)$ is a continuous function that has its own derivative the latter is called **the second derivative** of $f(x)$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

If the second derivative is continuous and differentiable we can find **the third derivative** of $f(x)$,

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \quad \textit{etc.}$$

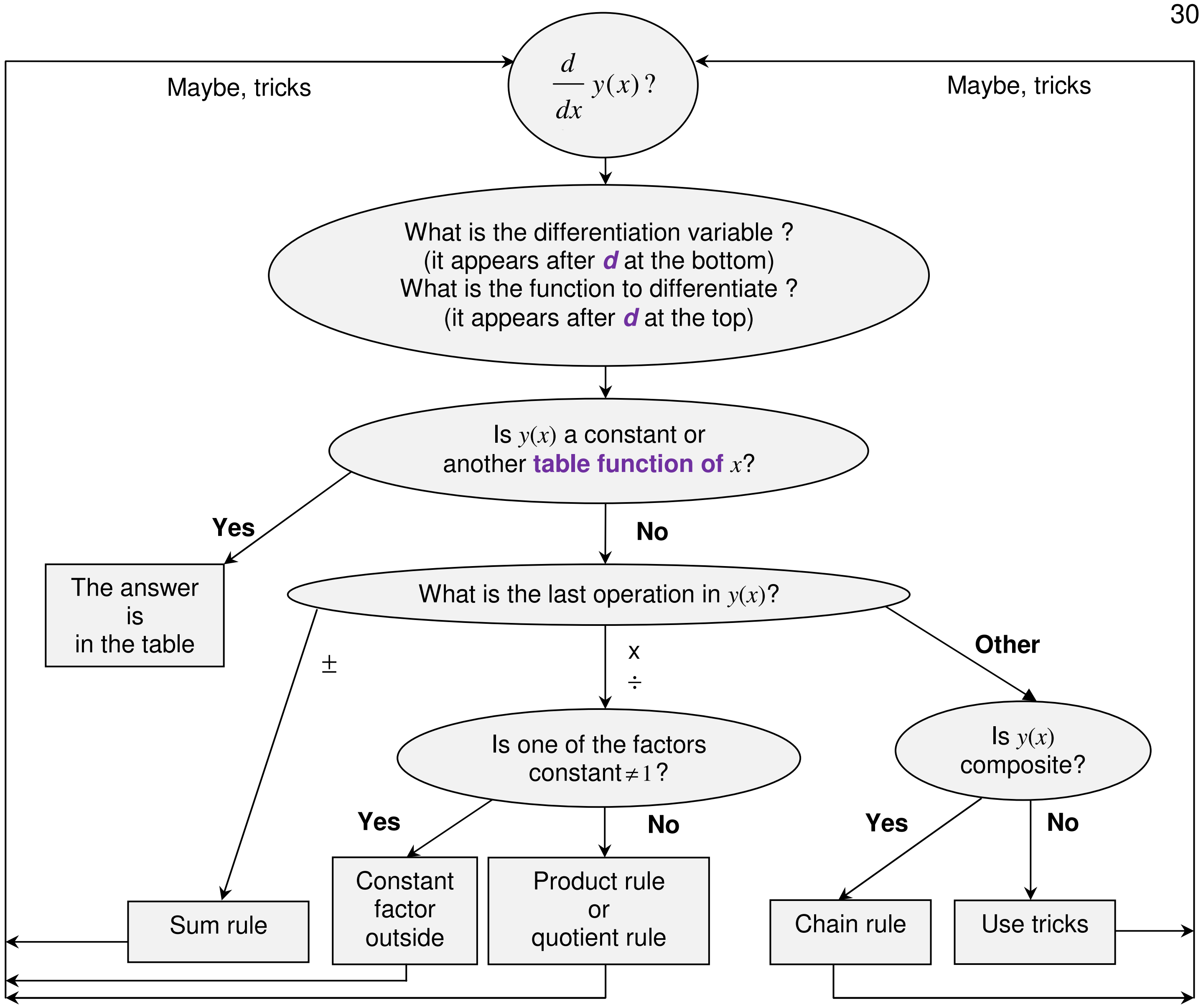


Figure 15.1. Decision Tree for Differentiation.

15.5 The partial derivatives

If we consider a function of say, two variables (two arguments) $f(x,y)$ its **partial derivatives** with respect to x and y are denoted by $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$, respectively.

15.6 Applications of differentiation

Later you will find that all simple scientific phenomena and engineering systems that you are going to study are described by differential equations that relate values of measured variables to their first or second derivatives. You will be also shown that derivatives and limits are helpful in sketching functions and thus illustrating or even predicting behaviour of the above measurable variables. Finally you will find out how derivatives can help to approximate functions, e.g. how your calculators calculate $\sin$, $\cos$ values *etc.*

15.7 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 4 and Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Lecture 13 and study Solutions to Exercises in Lecture 13 using the STUDY SKILLS Appendix**
- **Revise Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 15 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{t^2+1}}$

b) $\dfrac{d}{d\sqrt{t^2+1}}e^{\sqrt{x^2+1}}$

Q2. Find the derivatives

a) $\dfrac{d}{du}(u^2+1)\sin u$

b) $\dfrac{d}{du}\dfrac{u^2+1}{\sin u}$

Q3. Find the derivatives

a) $\dfrac{d\cos\sqrt{x^2+1}}{dx}$

b) $\dfrac{d\cos\sqrt{x^2+1}}{dy}$

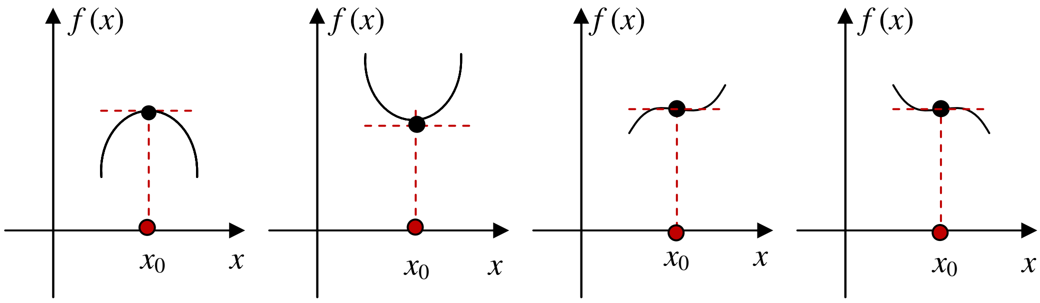
Q4. Differentiate y with respect to t if $y = (t+\sqrt{t^2+a^2})^n$.

Lecture 16. DIFFERENTIAL CALCULUS: Sketching Graphs Using Analysis

In general, any composite function can be sketched using our knowledge of limits and derivatives. In this Lecture we show how.

16.1 Stationary points

A **stationary point** is a point x_0 in the function domain where its derivative $f'(x_0) = 0$. There are four possible behaviours in the vicinity of the stationary point:



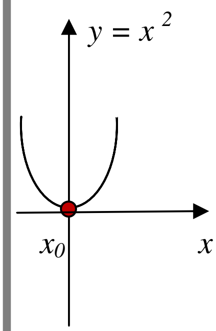
Examples: Sketch the following functions and find their stationary ponts x_0 :

1. $y = x^2$

$\frac{dy}{dx} = 0$

$2x = 0$

$x_0 = 0$

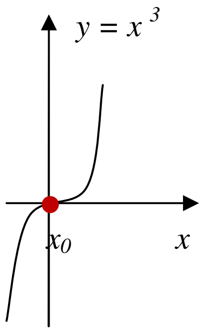


2. $y = x^3$

$\frac{dy}{dx} = 0$

$3x^2 = 0$

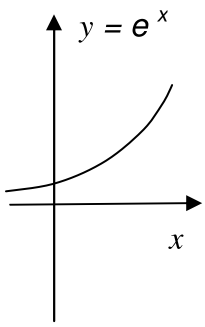
$x_0 = 0$



3. $y = e^x$

$\frac{dy}{dx} = e^x \neq 0$

No stationary points

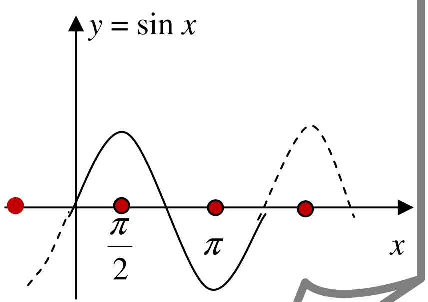


4. $y = \sin x$

$\frac{dy}{dx} = \cos x = 0$

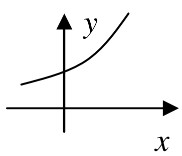
$x_0 = \frac{\pi}{2} n,$

n – odd integer
($n = 2m + 1$, m – integer)

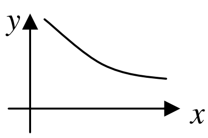


16.2 Increasing and decreasing functions

If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **increasing** on I.



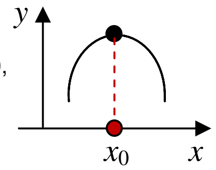
If derivative (local slope) $f'(x) > 0$ on interval I, function $f(x)$ is said to be **decreasing** on I.



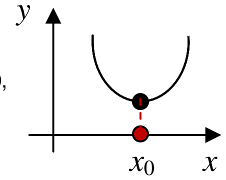
Examples: Functions $y = e^x$ and $y = \ln x$ are increasing everywhere; functions $y = |x|$ and $y = x^2$ are decreasing for $x < 0$ and increasing for $x > 0$.

16.3 Maxima and minima

If derivative (local slope) $f\,'(x) > 0$ to the left of x_0 and < 0 to the right of x_0, then function $f(x_0)$ is said to have a **maximum** at x_0.



If derivative (local slope) $f\,'(x) < 0$ to the left of x_0 and > 0 to the right of x_0, then function $f(x_0)$ is said to have a **minimum** at x_0.



Examples: $y = x^2$ has minimum at 0, x^3 has no minimum or maximum, $\sin x$ has maxima at $x = \frac{\pi}{2}m$ when m is odd and minima when m is even.

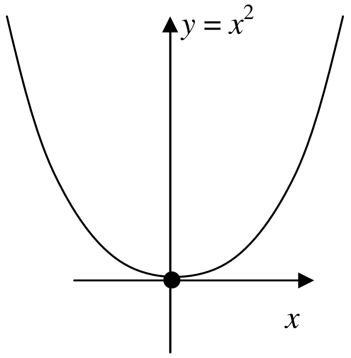
Examples:

1. Sketch $y = x^2 + 3x + 1$ by completing the square

Solution:

Completing the square, $y = (x + \frac{3}{2})^2 - \frac{9}{4} + 1 = (x + \frac{3}{2})^2 - \frac{5}{4}$.

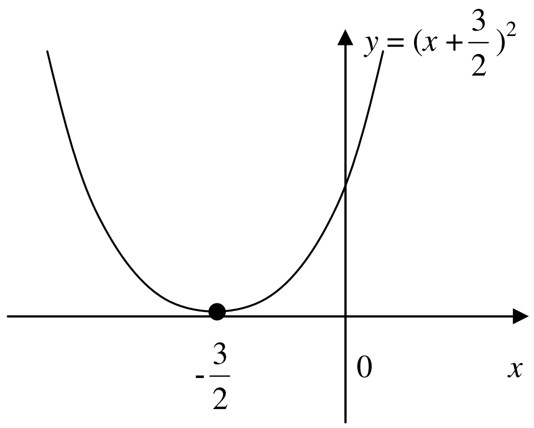
Step 1. Using the recipe for sketching by simple transformations, we first drop all constant factors and terms and sketch the basic shape of the function that remains:

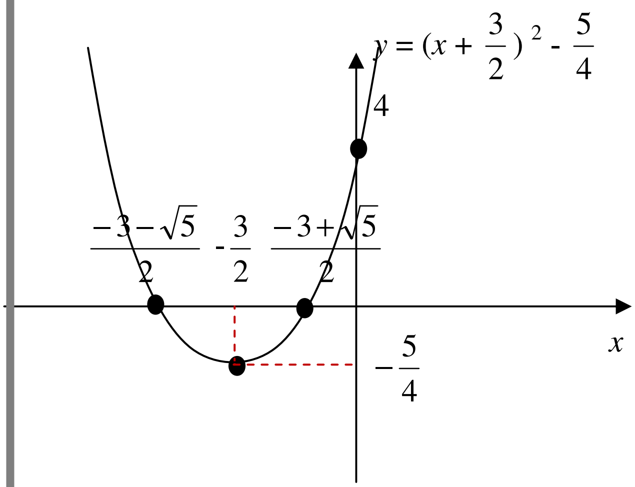


Step 2. We then reintroduce the constant factors and terms that appeared in the original functional equation one by one, in Order of Operations and sketch the resulting functions underneath one another:

$+\frac{3}{2}$ – constant term, first operation

$\Rightarrow$ translation wrt x-axis by $-\frac{3}{2}$





$-\frac{5}{4}$ – constant term, first operation

$\Rightarrow$ translation wrt y-axis by $-\frac{5}{4}$

Step 3.

Intersection with the y-axis: $x = 0, \ \ y = 4$

Intersection with the x-axis: $y = 0, \ \ x_{1,2} = -\frac{3 \pm \sqrt{9-4}}{2} = -\frac{3 \pm \sqrt{5}}{2} \approx -\frac{3 \pm 2.1}{2} \approx -2\frac{1}{2}, -\frac{1}{2}$

2. Sketch $y = x^2 + 3x + 1$ by analysis.

Solution

Step 1. **Intersection with the y-axis:** $x = 0, \ \ y = 4$

Intersection with the x-axis: $y = 0, \ \ x_{1,2} = -\frac{3 \pm \sqrt{9-4}}{2} = -\frac{3 \pm \sqrt{5}}{2} \approx -\frac{3 \pm 2.1}{2}$

Step 2. Find and sketch the first derivative

$$y' = 2x + 3$$

a) Find where the first derivative is zero: $2x + 3 = 0 \Rightarrow \ x = -\frac{3}{2}$

$\Rightarrow$ Stationary point of the function is $x = -\frac{3}{2}$

b) Find where the first derivative is positive and where it is negative:

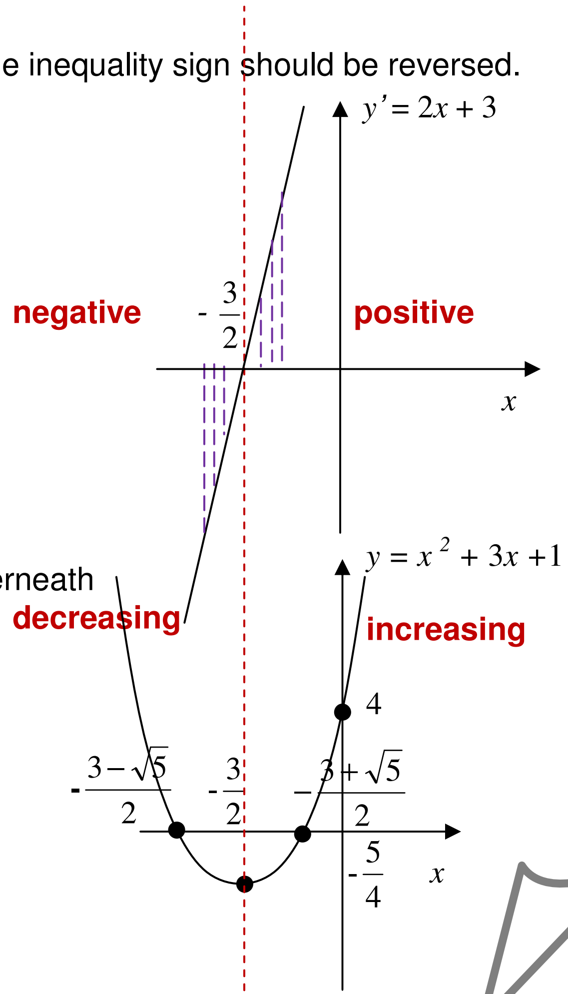
Can do so by solving the inequalities

$2x + 3 > 0 \ \ (y' > 0)$ for $x > -\frac{3}{2}$

$2x + 3 < 0 \ \ (y' < 0)$ for $x < -\frac{3}{2}$

Note: inequalities can be solved the same way as equations but when multiplying by negative factors the inequality sign should be reversed.

or graphically



Step 3: Sketch the function y directly underneath the first derivative y'

Step 4: Find the minimum function value

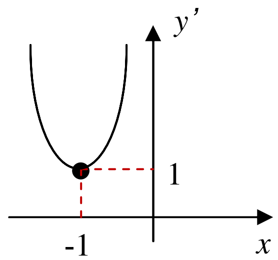
$$y\left(-\frac{3}{2}\right) = \frac{9}{4} - \frac{9}{2} + 1 = \frac{9-18+4}{4} = -\frac{5}{4}$$

3. Sketch $y = x^3 + 3x^2 + 4x + 1$ by analysis

Solution

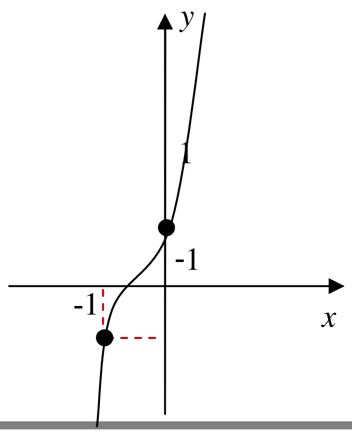
Step 1. Find and sketch the first derivative y'

$$y' = 3x^2 + 6x + 4 = 3\,(x^2 + 2x) + 4 = 3\,[(x+1)^2 - 1] + 4 = 3\,(x+1)^2 + 1 \;>\; 0$$



$y' > 0$ everywhere

Step 2. Sketch the function y underneath



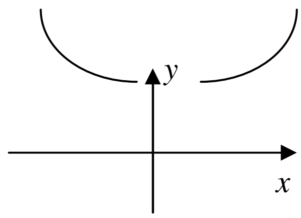
y is increasing everywhere

Special points: $y(0) = 1$

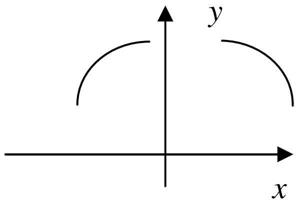
$y(-1) = -1$ (not that special – but easy to find)

Optional

16.5 Convex and concave functions



If the second derivative $f''(x) > 0$ on interval I the first derivative $f'(x)$ is increasing on I, and function $f(x)$ is said to be **convex** on interval I.



If the second derivative $f''(x) < 0$ on interval I the first derivative $f'(x)$ is decreasing on I and function $f(x)$ is said to be **concave** on interval I.

Examples: establish the regions of convexity of the following elementary functions:

$y = x^2$

$y' = 2x$

$y'' = 2 > 0$

$\Rightarrow\ y = x^2$ is convex everywhere

$y = x^3$

$y' = 3x^2$

$$y'' = 6x \begin{cases} > 0 \text{ if } x > 0 \\ < 0 \text{ if } x < 0 \end{cases}$$

$$\Rightarrow\ y = x^3 \text{ is } \begin{cases} \text{convex if } x > 0 \\ \text{concave if } x < 0 \end{cases}$$

$y = e^x$

$y' = e^x$

$y'' = e^x > 0$

$\Rightarrow\ y = e^x$ is convex everywhere

16.6 Inflexion points

If the second derivative $f''(x_0) = 0$ and $f''(x)$ changes sign at the stationary point x_0, then the stationary point x_0 is called **an inflexion point.**

Example: $y = x^3$ has an inflexion point at $x = 0$.

Thus, there is another way of determining whether the stationary point is at a maximum or minimum:

If $f'(x_0) = 0$ and $f''(x_0) > 0$, then the function is convex in the vicinity of x_0 and **x_0 is at a minimum.**

If $f'(x_0) = 0$ and $f''(x_0) < 0$, then the function is concave in the vicinity of x_0 and **x_0 is at a maximum.**

If $f'(x_0) = 0$ and $f''(x_0) = 0$, then ***further*** **checks are required.**

16.7 Sketching rational functions using analysis

We illustrate the general principles of sketching by analysis by applying the method to rational functions.

A **rational function** $R(x)$ is

$$R(x) = \frac{P(x)}{Q(x)},$$

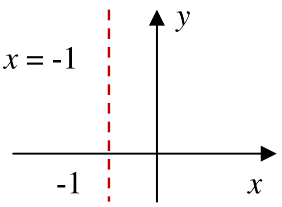
where *P* and *Q* are polynomials.

Examples:

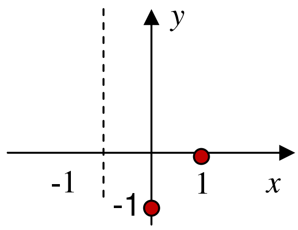
1. Sketch the rational function $y = \dfrac{x-1}{x+1}$

Solution

Step 1. Domain: $x \neq -1$ ($x = -1$ is called **a pole**, the dashed line $x = -1$ is called **a vertical asymptote**)

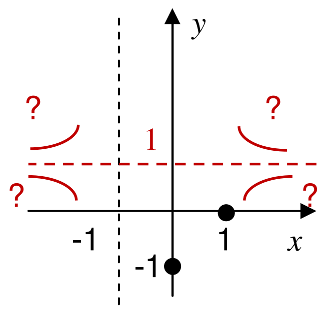


Step 2. Intersection with the y-axis: $x = 0$, $y = -1$
Intersection with the x-axis: $y = 0$, $x = 1$



Step 3. Behaviour at the domain boundaries:

$x \to \infty$, $y \to 1$
$x \to -\infty$, $y \to 1$

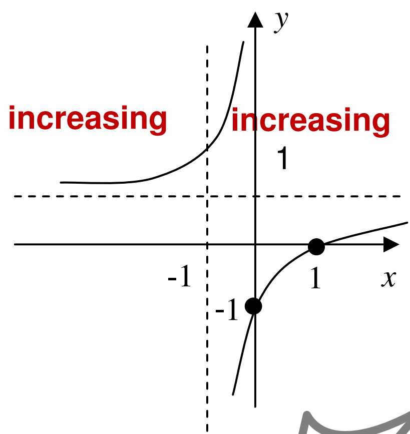


The dashed line $y=1$ is called **a horizontal asymptote**.
We know that the function approaches it when $|x|$ grows large but we do not know how.

Step 4. Find the first derivative:

$$\frac{dy}{dx} = \frac{x+1-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} > 0$$

$\Rightarrow$ y is increasing everywhere



2. Sketch $y = \dfrac{x-1}{x^2-4x+4}$

Solution

Step 1. Domain: $x \neq 2$ (x=2 is a pole, the line is a vertical asymptote)

Step 2. Intersection with the y-axis: $x = 0,\ \ y = -\dfrac{1}{4}$

Intersection with the x-axis: $y = 0,\ \ x = 1$

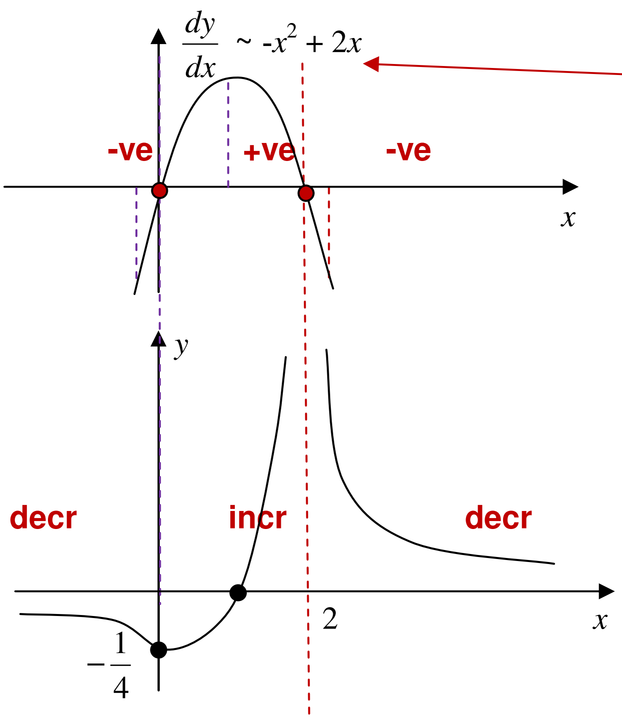
Step 3. Behaviour at the domain boundaries

$x \to \pm\infty,\ y \to \pm 0$

The line $y = 0$ is a **horizontal asymptote**

Step 4. Find the first derivative:

$$\frac{dy}{dx} = -\frac{x}{(x-2)^3} = \frac{-x^2+2x}{(x-2)^4} \begin{cases} = 0 & \text{at } x = 0 \\ > 0 \ (y \text{ is increasing}) & \text{for } 0 < x < 2 \\ < 0 \ (y \text{ is decreasing}) & \text{for } x < 0 \ \&\ \ x > 2 \end{cases}$$



This is not a derivative but it is 0, positive and negative where $\dfrac{dy}{dx}$ is – since $(x-2)^4$>0 when $x \neq 2$.

The sign $\sim$ means "behaves as".

Note that $x = 2$ turns the numerator $-x^2+2x$ into zero but $\dfrac{dy}{dx}$ is not zero at this point. The "phantom" zero appears because we multiplied both numerator and denominator of $\dfrac{dy}{dx}$ by $(x - 2)$. This was a trick performed to make sure that the denominator is never negative.

Optional

polynomial degree 2

Sketch y = $\frac{x^2+4x+2}{x+1}$ - **improper rational function**

polynomial degree 1

Difference in degrees = 1 $\Rightarrow$ the whole part is a **polynomial degree** 1

$$\frac{x^2+4x+2}{x+1} = Ax+B + \frac{C}{x+1}$$

polynomial degree 0

$\Uparrow$

polynomial degree 1

To find A, B and C use

1. **long division** or
2. **partial fractions method**

$$\frac{x^2+4x+2}{x+1} = Ax + B + \frac{C}{x+1} \Big/ (x+1)$$

$$x^2 + 4x + 2 = Ax(x + 1) + B(x + 1) + C$$

True for all x $\Rightarrow$ can choose any convenient values of x

1. Choose values that "kill" terms $x = -1$: $1 - 4 + 2 = C$, $C = -1$
   $x = 0$: $2 = B + C$, $2 = B - 1$, $B = 3$
2. Equate coefficients of the highest power: $1 = A$

$$y = x + 3 - \frac{1}{x+1}$$

Now we have to sketch $y = x + 3 - \frac{1}{x+1}$

Step 1. Domain: $x \neq -1$ ($x = -1$ is a pole)

Step 2. Intersection with the y-axis: $x = 0$, $y = 2$

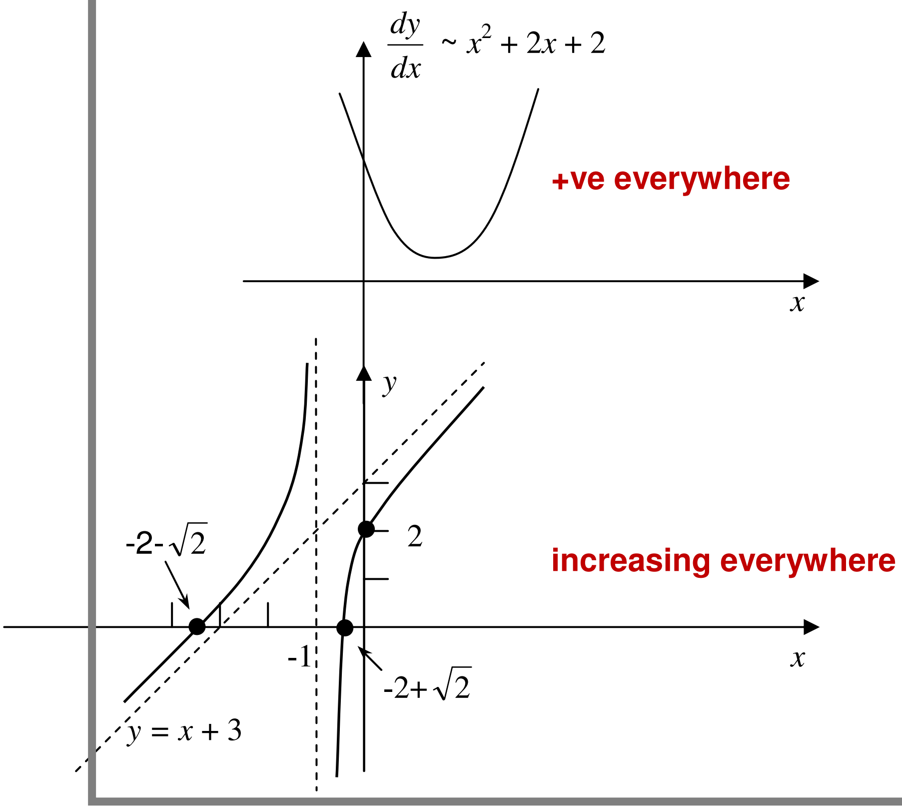
Intersection with the x-axis: $y = 0$, $x^2 + 4x + 2 = 0$, $x_{1,2} = -2 \pm \sqrt{2}$

Step 3. Behaviour at the domain boundaries:

$x \rightarrow \pm\infty$, $y \rightarrow x + 3$ – **an oblique asymptote** (typical for improper rational fractions)

Step 4. Find the first derivative:

$\dfrac{dy}{dx} = \dfrac{x^2+2x+2}{(x+1)^2} > 0$ because $x^2 + 2x + 2 = 0$ at $x_{1,2} = -1 \pm j$ (no real roots)



16.8 Applications of rational functions and their graphs

In control theory, single input single output (SISO) linear dynamic systems are often characterised by the so-called transfer functions. Any such transfer function is a rational function of one complex variable. Sketching magnitudes and phases of these functions helps design efficient control systems.

16.9 Instructions for self-study

- **Revise Lecture 9 and study Solutions to Exercises in Lecture 9 (sketching by simple transformations)**
- **Revise ALGEBRA Summary (addition and multiplication, factorising and smile rule, flip rule)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lectures 13 - 15 (limits and differentiation) and study Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**
- **Study Lecture 16 using the STUDY SKILLS Summary**
- **Do the following exercises:**

Q1. Sketch using analysis

a) $y = -3\,(x^3 - 3x^2 + x)$

b) $y = -\frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{8}$

Q2. Sketch $y = \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q3. Sketch $y = u(x+2)\ \dfrac{x^2 - 5x + 6}{x^2 + 1}$

Q4. Sketch $y = \dfrac{x^2 - 5x + 6}{x + 1}$

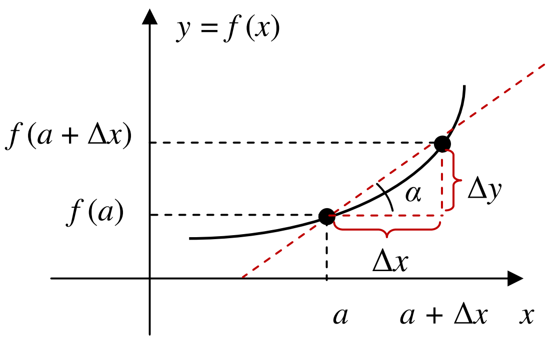
Lecture 17. Application of DIFFERENTIAL CALCULUS to Approximation of Functions: the Taylor and Maclaurin Series

17.1 Approximating a real function of a real variable using its first derivative

Let us discuss how we can **approximate** the value of function $f(x)$ at $a+ \Delta x$ if we know its value at a. It is clear from the figure below that

$$f(a+\Delta x) \approx f(a) + \Delta y .$$

How can we approximate Δy? To see this, let us revise the definition of the first derivative given in Lecture 13.



Question: What is the geometrical interpretation of derivative?
Answer: It is a local slope, $f'(x) = \tan\alpha$
Question: How can we find its approximate value?
Answer: $\tan\alpha \approx \dfrac{\Delta y}{\Delta x}$ when Δx is small.
Question: How to solve this approximate equality for Δy?
Answer: Multiplying both sides by Δx we can see that $\Delta y \approx \tan\alpha\, \Delta x = f'(x)\Delta x$

$$\Rightarrow f(a+\Delta x) \approx f(a) + f''(a)\, \Delta x$$

Verbalise: The function value near x **approximately equals** the function value at x + derivative at x times distance to x.

This is an approximation linear in Δx (in the vicinity of x the graph of $f(x)$ looks like a straight line). Sometimes we need approximations based on quadratic, cubic … *n-th* degree polynomials. Before showing how we introduce another way to represent a polynomial.

17.2 The Maclaurin polynomials

Question: What is a polynomial?
Answer:

Question: Conventionally we write the first term of the polynomial $P_n(x)$ as $a_n x^n$. All coefficients are constant with respect to x **and are denoted by letter** a. What will be the next term and the next and the next…?
Answer:

Any polynomial can be re-written in another form as

the Maclaurin polynomial: $P_n(x) = P_n(0) + P_n'(0)x + \dfrac{P_n''(0)}{2!}x^2 + \dfrac{P_n'''(0)}{3!}x^3 + \ldots \dfrac{P_n^{(n)}(0)}{n!}x^n$

where n **factorial** $n! = 1\times2\times3\times4\times\ldots\times n$.

We can show this using the following steps:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 \quad \Rightarrow \quad P_n\ (0) = a_0 \quad \Rightarrow \quad a_0 = P_n\ (0)$$

$$P_n'(x) = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + 3a_3 x^2 + 2a_2 x \ + a_1 \quad \Rightarrow \quad P_n\ (0) = a_1 \quad \Rightarrow \quad a_1 = P_n'\ (0)$$

$$P_n''(x) = a_n n(n-1)x^{n-2} + a_{n-1}(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 x \ + 2a_2 \Rightarrow P_n''(0) = 2a_2 \Rightarrow a_2 = \frac{P_n''(0)}{2}$$

$$P_n'''(x) = a_n n(n-1)(n-2)x^{n-3} + ... + 3\cdot 2\cdot a_3 \quad \Rightarrow \quad P_n'''(0) = 2\cdot 3\ a_3 \Rightarrow a_3 = \frac{P_n'''(0)}{3!}$$

$\dots \qquad \dots$

$$P_n^{(n)}(x) = a_n n(n-1)(n-2)....[n-(n-1)]x^{n-n} \quad \Rightarrow \quad P^{(n)}(0) = n!\ a_n \quad \Rightarrow \quad a_n = \frac{P^{(n)}(0)}{n!}$$

$$\Rightarrow P_n(x) = P_n(0) + P_n'(0)x + \frac{P_n''(0)}{2!}x^2 + \frac{P'''(0)}{3!}x^3 + ...\frac{P_n^{(n)}(0)}{n!}x^n$$

Verbalise: The Maclaurin polynomial at x is the polynomial at 0 + 1st derivative at 0 times x + 2nd derivative at 0 times x squared over $2!$ + ….

17.3 The Taylor polynomials

In the same way we can prove that any polynomial can be re-written as

the Taylor polynomial: $P_n(x) = P_n(a) + P_n'(a)(x-a) + ...\dfrac{P_n^{(n)}(a)}{n!}(x-a)^n$

Verbalise: The Taylor polynomial at x is the polynomial at a + derivative at a times the signed distance from x to a + ….

Note: in the Maclaurin series x can be viewed as the signed distance from x to 0.

17.4 The Taylor series

Many differentiable functions $f(x)$ can be represented using the infinite

Taylor series: $f(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-2)^2 + ...$

Verbalise: The function value at x is the function value at a + derivative at a times the distance to a + ….

These functions can be **approximated** as polynomials using **truncated Taylor series** (that is, Taylor polynomials).

Consider the approximation error **Optional**

$$e_n\,(a, x) \equiv f(x) - P_n\,(a, x)$$

If the error $e_n\,(x) \to 0$ as $n \to \infty$, then $f\,(x)$ equals its Taylor series at x and $y(x) \approx P_n(a, x)$ is an approximation based on the n-th order Taylor polynomial.

For e^x, $\sin x$, $\cos x$, $e_n\,(x - a) \to 0$ faster than $(x - a)^n$

Proving the above results and checking the limiting behaviour of the error e_n is rather involved.

17.5 The Maclaurin series

If $a = 0$, the Taylor series is called the **Maclaurin series.**

Examples:

1. Find the Maclaurin series for e^x.

Solution

Step 1. The general Maclaurin series is

$$f\,(x) = f\,(0) + f\,'(0)x + \frac{f\,''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f\,(x) = e^x$, $f\,'(x) = e^x$, $f\,''(x) = e^x$ $\ldots$ $f^{(n)}(x) = e^x$	$f\,'(0) = 1$ $f\,''(0) = 1$ $f^{(n)}(0) = 1$

Step 4. Substitute the above values into the general the Maclarin series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\ \ldots$$

$\Rightarrow e^x \approx 1 + x, |x| < 1$ (in the vicinity of 0 the graph of e^x is almost a straight line)

2. Find the Maclaurin series for $\sin x$.

Solution

Step 1. The general Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f'(x) = \cos x$ $f''(x) = -\sin x$ $f'''(x) = -\cos x$ $f^{(IV)}(x) = \sin x$ $\ldots$	$f'(0)=1$ $f''(0)=0$ $f^{(IV)}(0)=0$

Step 4. Substitute the above values into the general Maclarin series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$\Rightarrow \sin x \approx x, |x| < 1$ (in the vicinity of 0 the graph of $\sin x$ is almost a straight line)

c) Find the Maclaurin series for $\cos x$.

Solution:

Step 1. $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \ldots$

Step 2. Function and its derivatives are	**Step 3.** Values of the function and its derivative at 0 are
$f(x) = \cos x$ $f'(x) = -\sin x$ $f''(x) = -\cos x$	$f(0) = 1$ $f'(0) = 0$ $f''(0) = -1$

Step 4.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \quad -$$

$\Rightarrow \cos x \approx 1 - \dfrac{x^2}{2}, \; |x| < 1$ (in the vicinity of 0 the graph of $\cos x$ looks like a parabola – reflected **wrt** (with respect to) the horizontal axis shifted up by 1).

Thus, while any polynomial can be **re-written** as a Taylor or MacLaurin polynomial, many other n time differentiable functions can be **approximated** by Taylor or MacLaurin polynomials $P_n(x)$. The discussion on whether such approximation is possible or how to decide on the optimal degree of the approximating polynomial lies outside the scope of these notes.

17.6 L'Hospital's rule

Let us find $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ when $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)=0$, so that we have indeterminacy $\dfrac{0}{0}$.

In this case we can write

$$:(x - a)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(a)(x-a)+\ldots}{g'(a)(x-a)+\ldots}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

The recipe

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$$

is called **L'Hospital's** rule.

If still get the $\dfrac{0}{0}$ indeterminacy – continue applying the rule (find the second derivatives of the numerator and denominator, their third derivatives *etc.*)

Examples:

1. Find $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$.

Solution: $\lim\limits_{x\to 0} x=\lim\limits_{x\to 0}\sin x=0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\cos x}{1}=1$

2. Find $\lim\limits_{x\to 0}\dfrac{1-e^x}{x}$.

Solution: $\lim\limits_{x\to 0} x=\lim\limits_{x\to 0} 1-e^x=0$. Applying L'Hospital's rule, $\lim\limits_{x\to 0}\dfrac{1-e^x}{x}=\lim\limits_{x\to 0}\dfrac{-e^x}{1}=-1$

17.7 Applications

The above series are used by calculators to give approximate values of elementary functions, $\sin$, $\cos$, exponent *etc.*

17.8 Instructions for self-study

- **Revise ALGEBRA Summary (smile rule)**
- **Revise Lecture 5 (polynomials)**
- **Revise FUNCTIONS Summary**
- **Revise Lectures 13 - 14 (limits)**
- **Revise Lectures 14 - 15 (differentiation) and Solutions to Exercises in Lecture 14 using the STUDY SKILLS Appendix**

- **Revise Lecture 16 using the STUDY SKILLS Appendix**
- **Study Lecture 17 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1.

a) Find the first three non-zero terms in the Maclaurin series for $y=\dfrac{1}{1+x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\dfrac{1}{1+x}$ in the vicinity of $x=0$

c) Find, without using a calculator, the approximate value of $\dfrac{4}{5}$

d) Find $\lim\limits_{x\to -1}[\dfrac{1}{1+x}(1-e^{1+x})]$.

Q2.

a) Find the first three non-zero terms in the Maclaurin series for $y=\dfrac{1}{1-x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\dfrac{1}{1-x}$ in the vicinity of $x=0$.

c) Find, without using a calculator, the approximate value of $\dfrac{4}{3}$

d) Find $\lim\limits_{x\to 1}[\dfrac{1}{1-x}\sin(1-x)]$

Q3.

a) Find the first three non-zero terms in the Maclaurin series for $y=\sqrt{1+x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\sqrt{1+x}$ in the vicinity of $x=0$

c) Find, without using a calculator, the approximate value of $\sqrt{\dfrac{4}{3}}$

d) Find $\lim\limits_{x\to -1}\dfrac{\sqrt{1+x}}{1-x^2}$

Q4.

a) Find the first three non-zero terms in the Maclaurin series for $y=\sqrt{1-x}$

b) Use the first two terms of the above the Maclaurin series to approximate $y=\sqrt{1-x}$ in the vicinity of $x=0$

c) Find, without using a calculator, the approximate value of $\sqrt{\dfrac{4}{5}}$

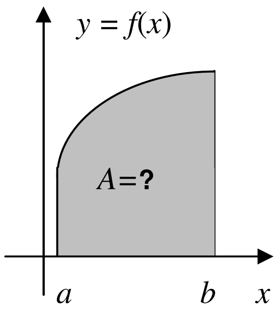
d) Find $\lim\limits_{x\to 1}\dfrac{\sqrt{1-x}}{1-x^2}$

Lecture 18. INTEGRAL CALCULUS: Integration of Real Functions of One Real Variable (Definite Integrals)

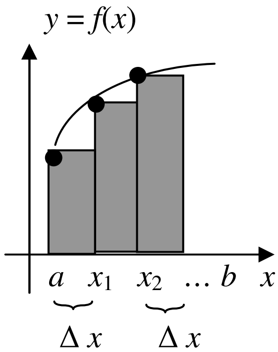
Integration (over a finite interval) of a function is a new advanced operation on functions. If it exists, the outcome is a number (or a constant with respect to the control variable) called **a definite integral**.

18.1 A definite integral

Consider the area A between a curve, which is the graph of a function $f(x)$, the horizontal axis and lines $x = a$ and $x = b$:



Area A between the curve and horizontal axis *approximately equals* the sum of areas of inscribed thin rectangles



Question: What is the area of first rectangle, second rectangle *etc.*?
Answer:

In other words, this area A approximately equals,

$$A \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + .. + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)$$

where $x_0 = a$ and $x_n = b.$

Question: How many rectangles do we have here?
Answer:

Taking the limit $\Delta x \to 0,$ so that the width of the rectangles gets smaller and smaller, the number of rectangles grows larger and larger ($n \to \infty$) and we can write that this area A exactly equals

if the limit exists

$$A = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\Delta x \equiv \int_a^b f(x)dx$$

Thus, we introduced a **definite of the definite integral** of the function $f(x)$ over the interval [*a, b*],

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

The **geometrical interpretation of the definite integral** of a real function of one real variable is a "(**signed**) area between a curve (the graph of this function) and horizontal axis". The area is **signed,** because on the interval where $f(x)$ is negative, all products $f(x_i)\,\Delta x$ are negative. Thus, when the horizontal axis is "above the curve" the integral is negative and equal the area between the curve and horizontal axis with the – sign.

Example: $\int_0^{2\pi} \sin x dx = ?$ **Answer:** 0 (sketch the graph of the sin-function to see that).

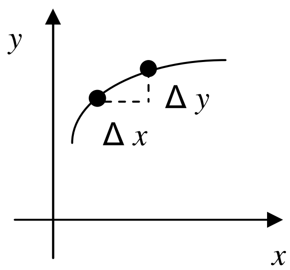
18.2 Notation

$\int$ - integral (an elongated letter S, to remind us that we are talking about the limit of a sum Σ)

$\int_a^b ...dx$ - a definite integral between a and b (to remind us that we are talking about the limit of a sum of areas of thin rectangles of width Δx fitted between the curve, horizontal axis and straight lines $x = a$ and $x = b$).

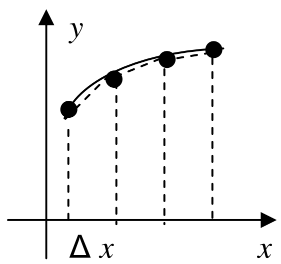
a, b are called **the lower and upper limits of integration**, respectively.

18.3 Discussion of definitions of a definite Integral and derivative



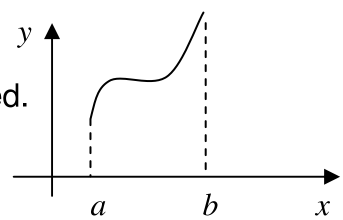
The derivative
approximately equals
the slope of local inscribed segment

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

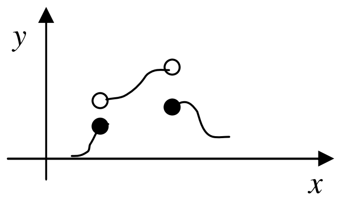


When defining integrals and derivatives we reduce a difficult problem involving a curve to many simple problems involving small straight segments. Ability to reduce complicated unfamiliar pictures to simple familiar patterns is an invaluable engineering skill.

18.4 Examples of integrable functions



1. Any $f(x)$ continuous in $[a, b]$ is **integrable**, that is, can be integrated.
   Question: Why?
   Answer:



2. Any bounded $f(x)$ with finite number of jumps can be integrated
   Question: Why?
   Answer:

Note: If we change $f(x)$ at a finite number of points its **integrability** does not change.

Question: Why?
Answer:

Revision

Let C be a condition.
Let E be an event.

If $C \Rightarrow E$, then the condition C is **sufficient** for event E to take place.
If $E \Rightarrow C$, then the condition C is **necessary** for event E to take place.

Question: Is continuity sufficient for integrability, necessary or both?
Answer:

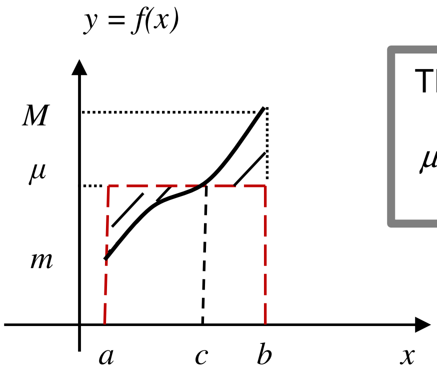
Note: Continuity is not sufficient for differentiability. Continuity is necessary for differentiability.

18.5 The Mean Value Theorem

If $f(x)$ is continuous, then there exists a value μ, such that $m \le \mu \le M$ and

$$\int_a^b f(x)dx = \mu(b-a),$$

where m is the smallest value of $f(x)$ on the interval $[a, b]$ and M is the largest value of $f(x)$ on the interval $[a, b]$. μ is the Greek letter *mu*.



The area under the curve (between a and b) equals $\mu(b-a)$, the area of a rectangle (in red, long dashes)

$\mu = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$ is called the **mean value** of $f(x)$ on the interval $[a, b]$.

Why is it called that?

1. The mean value of the sequence $\{1, 3\}$ is $\frac{1+3}{2}$

2. The mean value of the sequence $\{x_1, \ldots, x_n\}$ is $\frac{x_1 + \ldots + x_n}{n}$

3. The mean value of the sequence $\{f(x_1), \ldots, f(x_n)\}$ is $\frac{1}{n}\sum_{i=0}^{n-1} f(x_i)$

Question: With reference to Section 18.1, if n is the number of rectangles of width Δx used to approximate the area between $f(x)$ and the horizontal axis what is n, given a and b?
Answer:

Question: What is $\lim_{\Delta x \to 0} n\Delta x$?

Answer:

This means that we can write

$$\frac{1}{n}\sum_{i=0}^{n-1} f(x_i) = \frac{1}{n\Delta x}\sum_{i=0}^{n-1} f(x_i)\Delta x \underset{\Delta x \to 0}{\rightarrow} \frac{1}{b-a}\int_a^b f(x)dx$$

This is why $\mu = \frac{1}{b-a}\int_a^b f(x)dx$ is called the mean value of $f(x)$ on the interval $[a, b]$.

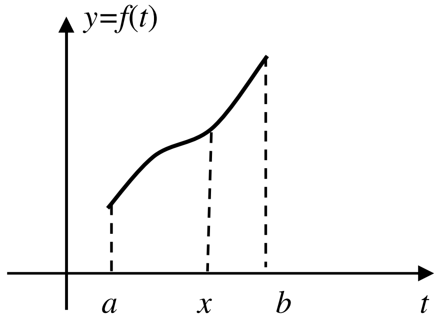
18.6 A definite integral with a variable upper limit - function $\Phi(x)$

Let $f(x)$ be integrable in $[a, b]$. We can introduce a new advanced operation on a function – evaluating a definite integral with a variable upper limit. The result is

$$\Phi(x) = \int_a^x f(t)dt,$$

a **function of** x, since if a is fixed for every x we have one and only one value for the signed area between the curve and horizontal axis.

Question: Does this area depend on t?
Answer:



18.7 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus : If $f(x)$ continuous, then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Attempt to prove it yourself (see Q2.)

18.8 Applications of integration

Later you will find that apart from being useful in finding areas and volumes, integration is necessary to solve differential equations that describe all scientific phenomena and engineering systems that you are going to study.

18.9 A historical note

According to Knowledgewiki, "integration can be traced as far back as ancient Egypt, *circa* 1800 BC, with the Moscow Mathematical Papyrus demonstrating knowledge of a formula for the volume of a pyramidal frustum. The first documented systematic technique capable of determining integrals is the method of exhaustion of Eudoxus (*circa* 370 BC), which sought to find areas and volumes by breaking them up into an infinite number of shapes for which the area or volume was known. This method was further developed and employed by Archimedes and used to calculate areas for parabolas and an approximation to the area of a circle. Similar methods were independently developed in China around the 3rd Century AD by Liu Hui, who used it to find the area of the circle. This method was later used in the 5th century by Zu Chongzhi to find the volume of a sphere. That same century, the Indian mathematician Aryabhata used a similar method in order to find the volume of a cube.

The next major step in integral calculus came in the 11th century, when the Iraqi mathematician, Ibn al-Haytham (known as *Alhazen* in Europe), devised what is now known as "Alhazen's problem", which leads to an equation of the fourth degree, in his *Book of Optics*. While solving this problem, he performed an integration in order to find the volume of a paraboloid. Using mathematical induction, he was able to generalize his result for the integrals of polynomials up to the fourth degree. He thus came close to finding a general formula for the integrals of polynomials, but he was not concerned with any polynomials higher than the fourth degree. Some ideas of integral calculus are also found in the *Siddhanta Shiromani*, a 13thcentury astronomy text by Indian mathematician Bhāskara II.

The next significant advances in integral calculus did not begin to appear until the 16th century. At this time the work of Cavalieri with his *method of indivisibles*, and work by Fermat, began to lay the foundations of modern calculus. Further steps were made in the early 17th century by Barrow and Torricelli, who provided the first hints of a connection between integration and differentiation.

The major advance in integration came in the 17th century with the independent discovery of the fundamental theorem of calculus by Newton and Leibniz. The theorem demonstrates a connection between integration and differentiation. This connection, combined with the comparative ease of differentiation, can be exploited to calculate integrals. In particular, the fundamental theorem of calculus allows one to solve a much broader class of problems. Equal in importance is the comprehensive mathematical framework that both Newton and Leibniz developed. Given the name infinitesimal calculus, it allowed for precise analysis of functions within continuous domains. This framework eventually became modern Calculus, whose notation for integrals is drawn directly from the work of Leibniz."

http://www.knowledgewiki.org/article/Integral?enk=B7FmqUapZqFmGSahZJFkkQaRhoFkiWahBsFGgQapBxkm#History

18.10 Instructions for self-study

- **Revise Lecture 16 and study Solutions to Exercise in Lecture 16 using the STUDY SKILLS Appendix**
- **Revise Lecture 17 using the STUDY SKILLS Appendix**
- **Study Lecture 18 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find using the first principles, that is using the definition of definite integral,

a) $\int_0^1 0.2dp$

b) $\int_0^1 5dq$

c) $\int_0^1 0.5du$

d) $\int_0^1 sds$

e) $\int_0^1 3udu$

f) $\int_0^1 4vdv$

Q2 **(advanced)**. Prove that if $f(x)$ continuous, then $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ (using the first principles, that is using the definition of definite integral, and Mean Value Theorem). Describe this formula in words.

Lecture 19. INTEGRAL CALCULUS: Integration of Real Functions of One Real Variable (Indefinite Integrals)

19.1 The relationship between differentiation and integration

According to the Fundamental Theorem of Calculus if $f(x)$ is continuous we have

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)\,. \qquad (19.1)$$

Question: What is the order of operations on $f(t)$ in the left hand side above? (treat "$\int \ldots dt$" as bracketing the integrand).
Answer:

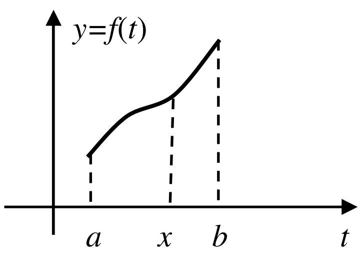
Thus, differentiation of a continuous function undoes what integration does, that is, the **differentiation is inverse to integration.** Let us now subdivide the $[a,\ x]$ interval into small subintervals with $x_0 = a$ and $x_n = x$:

$$\int_a^x \frac{d}{dt} f(t)dt = \lim_{\Delta x\to 0}\sum_{i=0}^{n-1}\frac{\Delta f(x_i)}{\Delta x}\Delta x = \lim_{\Delta x\to 0}[\Delta f(x_0)+\Delta f(x_1)+\ldots+\Delta f(x_{n-1})] = \qquad (19.2)$$

$$\lim_{\Delta x\to 0}\{[f(x_1)-f(x_0)]+[f(x_2)-f(x_1)]+\ldots+[f(x_n)-f(x_{n-1})]\} = f(x_n)-f(x_0)\equiv f(x)+c$$

Question: What is the order of operations on $f(t)$ in the left land side above?
Answer:

Thus, **integration is not a perfect inverse to differentiation**, the definite integral with the variable upper limit depends on the lower limit too. Another way to see this is as follows: Examining the graph below



and remembering that the integral can be interpreted as a signed area, we can write $\int_a^x f(t)dt = \int_{a_0}^x f(t)dt + \int_a^{a_0} f(t)dt \equiv \int_{a_0}^x f(t)dt + c$. Since $c = \int_a^{a_0} f(t)dt$ is **constant with respect to** x (though it depends on a_0), Eq. (19.1) implies that

$$\frac{d}{dx}(\Phi(x,a_0)+c) = f(x)\,,$$

where instead of $\Phi(x)$ (the notation used in the last Lecture), we introduce a more logical notation $\Phi(x,a) = \int_a^x f(t)dt$ (a definite integral with a variable upper limit x actually depends on the lower limit a too).

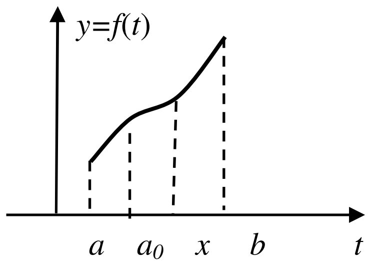
This means each function $f(x)$ has infinitely many definite integrals $\Phi(x,a)$. For this reason, the integral $\Phi(x,a)$ is called an **antiderivative** of $f(x)$, rather than the *inverse derivative* of $f(x)$ (remember, an inverse function – as any function – is supposed to assign only one value to any value of its argument).

19.2 An indefinite integral

We have shown that for every continuous function there are infinitely many antiderivatives, all differing from one another by a constant term. Often instead of the antiderivative we use a somewhat confusing shorthand

the **indefinite integral** $F(x) \equiv \int f(x)dx \equiv \int_a^x f(t)dt$ (19.3)

Using the graph below



and the geometrical interpretation of the definite integral as a signed area, we can write

$$\int_a^x f(t)dt = \int_{a_0}^x f(t)dt + \int_a^{a_0} f(t)dt$$

Substituting the above decomposition into definition (19.3) we get

$$\int f(x)dx \equiv \int_{a_0}^x f(t)dt + c \quad (19.4)$$

where $c = -\int_a^{a_0} f(t)dt$ is a constant with respect to x. This is a reiteration of the fact that the antiderivative is **defined up to a constant term**, because we do not know (an do not need to know) what is its lower limit of integration

We can now introduce a very important formula for evaluating definite integrals:

$$\int_a^b f(x)dx = F(b) - F(a) \quad (19.5)$$

The above formula implies that

1. $\int_b^a f(x)dx = -\int_a^b f(x)dx$ **(Why?)**

2. if we know an indefinite integral we can **evaluate** (find the value of) any associated definite integral (note that **the c-terms** in $F(b)$ and $F(a)$ in Eq. (19.5) **cancel out**).
Finally, note that using the indefinite integral notation the two Fundamental Theorems of Calculus (19.1) and (19.2) can be re-written as

$$\frac{d}{dx}\int f(x)dx = f(x)\,, \qquad \int \frac{df(x)}{dx}dx = f(x) + c \tag{19.6}$$

Terminology

$f(x)$ – integrand, x - integration variable, $\int ...dx$ is an indefinite integral with respect to x. It **has** limits but these limits are allowed to vary, that is, are indefinite.)

19.3 Finding an indefinite integral

To find an indefinite integral we use the **Integration Table** of elementary functions, **Rules** (or **Methods**) for integrating combinations of functions and **Decision Tree** to decide which rule or Table entry to use.

19.4 The Integration Table

We can derive Integration Table for elementary functions using Differentiation Table.

Differentiation Table

$g(x)$	$\frac{dg(x)}{dx}$
constant	0
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
x^k	kx^{k-1}
$\ln x$, $x>0$	$\frac{1}{x}, x \neq 0$

$\Rightarrow$

Integration Table

integrand **integration variable**

$f(x)$	$F(x) = \int f(x)dx$
0	c (constant)
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}, x \neq 0$	$\ln\lvert x\rvert + c$

Let us check that $\int \frac{1}{x}dx = \ln|x|$, for any $x \neq 0$. Indeed, **Optional**

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\Rightarrow \frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x}, & x > 0, \\ -(-\frac{1}{x}), & x < 0, \end{cases}$$ where we used the chain rule of differentiation.

Examples:

1. $\int e^{x^2} dx^2 = e^{x^2} + c$
2. $\int \sin e^w de^w = -\cos e^w + c$
3. $\int dx = \int 1dx = \int x^0 dx = x + c \quad \Rightarrow \quad \int dx = \int \frac{dx(t)}{dt} dt = x + c$

19.5 Elementary integration rules

1. $\int \alpha f(x)dx = \alpha \int f(x)dx$ - **constant factor out**
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ - **sum rule**

linearity of integration

These rules can be proven using the definition of integral and rules for limits.

Examples:

1. Integrate $\int 2\sin x\, dx$

Solution
Integration variable – x
Integrand - $2\sin x$
It is not a table function of x
The last operation in the integrand – multiplicaton
One of the factors – 2 - is a constant
Use the "constant factor out" rule:
Constant factor – $\alpha = 2$, variable factor term $f(x) = \sin x$
Substituting α and $f(x)$ nto the "constant factot out" rule:
$\int 2\sin x\, dx = 2\int \sin x\, dx = -2\cos x + c$

2. Integrate $\int (\sin x + \cos x)dx$

Solution
Integration variable – x
Integrand - $\sin x + \cos x$
It is not a table function of x
The last operation in the integrand – addition
Use the "separate terms" rule:
First term $f(x) = \sin x$, second term $g(x) = \cos x$
Substituting $f(x)$ and $g(x)$ into the "separate terms" rule:
$\int (\sin x + \cos x)dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + c_1 + \sin x + c_2 = \sin x - \cos x + c$

19.6 Integration Decision Tree

Similarly to the Decision Tree for Limits and Decision Tree for Differentiation, the Decision Tree for Integration (figure 19.1) allows you to decide at each step which Integration Rule or Integration Table entry to use when finding an indefinite integral. The decision depends on the last operation in the integrand. Do not forget to start at the top of the Tree and then follow the arrows that are associated with the correct answers (if any). After you have gone through the Decision Tree once check whether the operation of integration has been carried out. If not, hopefully, now you have simpler functions to integrate. In order to do that go over the Decision Tree again, substituting new integrands for $f(x)$ (and your integration variable for x).

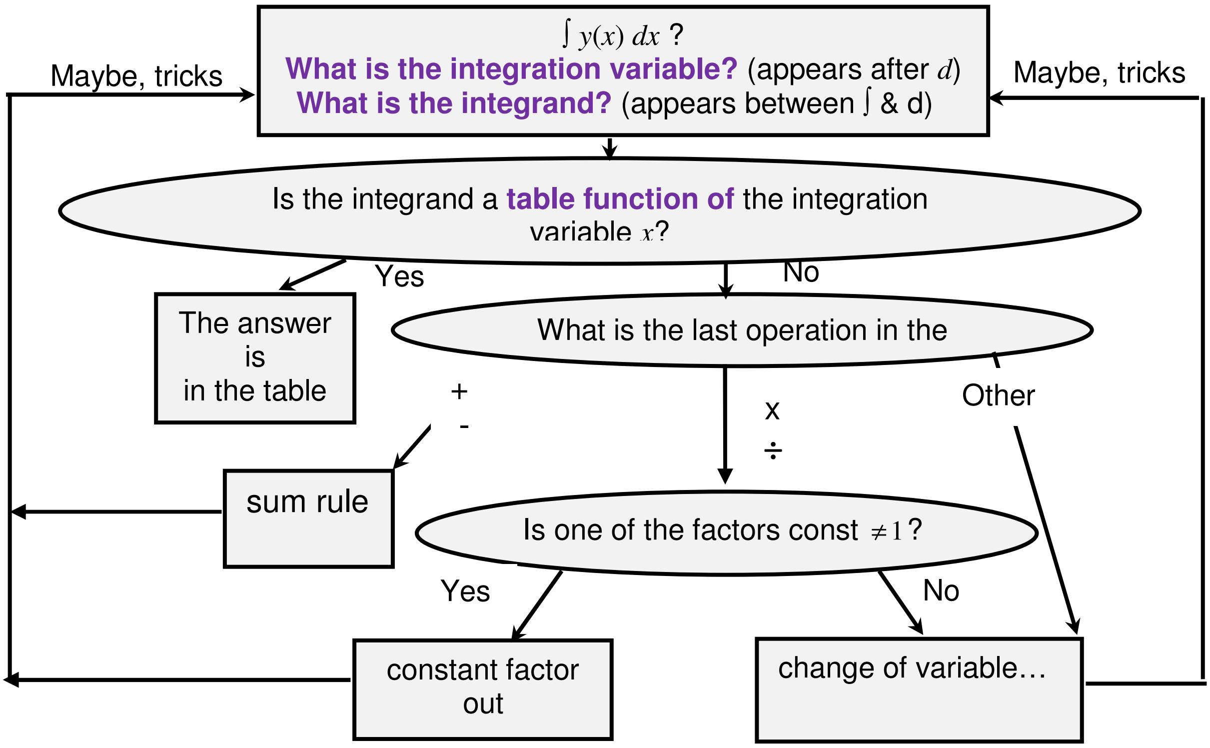


Figure 19.1. Decision Tree for Integration (incomplete).

19.7 Integration method (rule) of change of variable (substitution)

There are many other integration rules apart from the elementary ones considered in Section 19.4. We will study them later. They all follow from the rules for differentiation but are more complicated. First we introduce the simplest of them. It is called **the change of variable method**.

Examples:

1. $\int (3x+1)^{2.7}\,dx = \int u^{2.7}\frac{du}{3} = \frac{1}{3}\frac{u^{3.7}}{3.7} + c = \frac{1}{11.1}(3x+1)^{3.7} + c$

Integration variable – x
Integrand – $(3x + 1)^{2.7}$
The integrand is **not** a **table function of** x.
The last operation in the integrand is power, hence the elementary rules do not apply.
Introduce a new variable that lumps together the first few operations in the integrand

$$u = 3x + 1 \,\Big/\, \frac{d}{dx}, \quad \frac{du}{dx} = 3 \,\Big/\, dx, \ \div 3, \qquad dx = \frac{du}{3}$$

Note: $\frac{du}{dx}$ is not an ordinary fraction. It is a limit of a fraction. Since the limit of a fraction is a fraction of limits it can be proven that **we can formally "multiply"** $\frac{du}{dx}$ **by** dx ***to*** **get** du**.**

Question: What is du equal to?
Answer:
Question: What is dx equal to?
Answer:

2. $\displaystyle\int_2^3 t\sin t^2 dt = -\frac{1}{2}\cos t^2\Big|_2^3 = -\frac{1}{2}(\cos 9 - \cos 4) \approx 0.129$

$$\int t\sin t^2 dt = \frac{1}{2}\int \sin u du = -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos t^2 + c$$

$$u = t^2, \frac{du}{dt} = 2t \,\Big/\, dt, \ tdt = \frac{du}{2}$$

3. $\displaystyle\int_1^2 \sin t\cos^2 tdt = -\frac{1}{3}(\cos^3 2 - \cos^3 1) \approx 0.0766$

$$\int \sin t\cos^2 tdt = -\int u^2 du = -\frac{u^3}{3} + c = -\frac{\cos^3 t}{3} + c$$

$u = \cos t,\ du = -\sin t\ dt$

4. $\int \frac{e^{\tan x}}{\cos^2 x} dx = \int e^u du = e^u + c = e^{\tan x} + c$

$u = \tan x,\ \frac{du}{dx} = \frac{1}{\cos^2 x},\ du = \frac{1}{\cos^2 x} dx,$

5. $\int \frac{3x^2+1}{x^3+x+2} dx = \int \frac{du}{u} = \ln\left|x^3+x+2\right| + c$

$u = x^3 + x + 2,\ du = (3x^2 + 1)\ dx$

6. $\int \frac{4}{5x-7} dx = 4\int \frac{dx}{5x-7} = \frac{4}{5}\int \frac{du}{u} = \frac{4}{5}\ln|5x-7| + c$

$u = 5x - 7,\ du = 5\ dx$

7. $\int \frac{t}{t^2+1} dt = \frac{1}{2}\ln\left(t^2+1\right) + c$

8. $\int \frac{e^{\frac{t}{2}}}{e^{\frac{t}{2}}+1} dt = 2\ln\left(e^{\frac{t}{2}}+1\right) + c$

19.8 Instructions for self-study

- **Revise ALGEBRA Summary (particularly, the words term, sum, factor, product)**
- **Revise Lecture 4 and Solutions to Solutions to Exercises in Lecture 4 (particularly, the operation of composition)**
- **Revise Summaries on the ORDER OF OPERATIONS and FUNCTIONS**
- **Revise Lecture 17 and study Solutions to Exercises in Lecture 17 using the STUDY SKILLS Appendix**
- **Revise Lecture 18 using the STUDY SKILLS Appendix**
- **Study Lecture 19 using the STUDY SKILLS Appendix**
- **Attempt the following exercises:**

Q1. Find

a) $\int \frac{1}{u} du$

b) $\int \frac{1}{u} dx$

c) $\int \frac{1}{u+t} d(u+t)$

d) $\int e^{\sin t} d \sin t$

e) $\int e^{n \sin t} de^{\sin t}$

Q2. Find

a) $\int (\frac{1}{u} + \cos u) du$

b) $\int (u + u^{2}) du$

c) $\int (u + 2u^{2} + u^3) du$

d) $\int (2v + 3v^{2} + 5u^5) dv$

Q3. Find

a) $\int \cos t \, e^{\sin t} dt$

b) $\int \frac{4}{3-t} dt$

c) $\int_{-1}^{1} \frac{2+6v}{2v+3v^2+5u^5} dv$

Additional Exercises

Q4. Use the change of variable (substitution) to find

a) $\int (4x+1)^7 dx$

b) $\int t^2 \sin(t^3+1) dt$ Hint: let $u = t^3 + 1$

c) $\int \sin(3x-1) dx$

d) $\int e^{2x-3} dx$

e) $\int x(2x^2+7)^4 dx$

f) $\int \sin^2 4t \cos 4t dt$

Q5 Evaluate

a) $\int_3^4 \frac{t}{\sqrt{t^2-3}}\,dt$

b) $\int_0^2 \frac{\ln z}{z}\,dz$. Hint: let $u = \ln z$

Q6. Find

a) $\int \frac{3x^2+2}{x^3+2x+5}\,dx$

b) $\int_2^3 t^2 \sin t^3\,dt$

c) $\int \frac{1}{u-1}\,du$

d) $\int \frac{1}{1-u}\,du$

Lecture 20. INTEGRAL CALCULUS: Advanced Integration Methods

In Lecture 19 we have introduced the table of indefinite integrals of elementary functions and two simple linearity rules: one for integrating a sum of two functions and one for integrating a product with one of the factors constant. We now introduce additional rules (or methods) for integrating products and quotients of elementary functions.

20.1 Integration of products of trigonometric functions

Examples:

1. $\int \cos^2 t \ dt =$

Question: Is $\cos^2 t$ a product?

Answer: No, but it can be easily re-written as a product, $\cos^2 t = \cos t \times \cos t$. The square of a cosine is simply connected to the cosine of a double angle

$$\cos 2t = \cos^2 t - \sin^2 t,$$

see the **Trigonometry Summary**.

We can eliminate $\sin^2 t$ above using **Pythagoras Theorem** in terms of angles.
Question: What is **Pythagoras Theorem** in terms of angles?

Answer: $\sin^2 t + \cos^2 t = 1$.

$$\Rightarrow \sin^2 t = 1 - \cos^2 t$$

Substituting $\sin^2 t$ into the **double angle identity** above gives us

$$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - (1 - \cos^2 t) = \cos^2 t - 1 + \cos^2 t = 2\cos^2 t - 1.$$

We can now use the **Decision Tree for Solving Simple Equations** to express $\cos^2 t$ in terms of $\cos 2t$, a simpler function of t,

$$\cos^2 t = \frac{1 + \cos 2t}{2}.$$

The latter expression involves simple operations (on $\cos 2t$) of addition and division by a constant, so we can integrate using the **linearity rules.**

$$\int \frac{1+\cos 2t}{2} dt = \frac{1}{2}\int (1 + \cos 2t) dt = \frac{1}{2}(\int dt + \int \cos 2t dt) = \frac{1}{2}(t + \int \cos 2t dt) =$$

The last integral is not in the **Integration Table** for elementary functions introduced in Lecture 17, but it can be found in Integration Tables in other textbooks or on internet. The integral can also be evaluated using the Integration Table for elementary functions and **change of variable method** presented in Lecture 17, with $u = 2t$. Whatever the method is used,

$$\int \cos 2t = \frac{\sin 2t}{2} + c.$$

Question: What is a derivative of $\cos 2t$?
Answer: $-2\sin 2t$
Verbalise: Thus, when differentiating $\cos$, a constant factor in the argument becomes a factor in the result and when integrating $\cos$, it appears in the denominator of the result.

$$\frac{1}{2}(t+\frac{\sin 2t}{2})+c=\frac{1}{2}t+\frac{\sin 2t}{4}+c$$

using Pythagoras Theorem

2. $\int \sin^2 t\,dt = \int (1-\cos^2 t)\,dt = \int 1dt - \int \cos^2 t\,dt = \frac{t}{2}-\frac{\sin 2t}{4}+c$, where we used the result obtained in Example 1.

3. $\int \sin mt \sin nt dt = \frac{1}{2}\int [\cos(m-n)t - \cos(m+n)t]dt = \frac{1}{2}[\frac{\sin(m-n)t}{m-n}-\frac{\sin(m+n)t}{m+n}]+c,$

where we used a trig identity for a product of sinuses (which can be obtained by combining the identity for cos of a difference, $\cos(A - B)$ and identity for cos of a sum, $\cos(A + B)$.

20.2 Integration by parts (integration of products of different types of functions)

Optional

The product rule for differentiation gives us a product rule for integration (traditionally called **integration by parts**)

$$\frac{d(u\cdot v)}{dx}=u\frac{dv}{dx}+v\frac{du}{dx} \Big/ dx$$

Question: What is dx?
Answer:

Multiplication by dx is purely formal, its meaning can be clarified by using properties of limits

$\Rightarrow d(u\,v) = u\,dv + v\,du$

$\Rightarrow u\,dv = d(u\,v) - v\,du \Big/ \int \quad \Rightarrow$

Integration by Parts formula: $\int u dv = uv - \int v du$

Examples:

1. $\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + c$

u - line: $u = x,\ \frac{du}{dx}=1,\ du = dx$

v – line: $dv = \cos dx \Big/ \int,\ v = \int \cos x\,dx = \sin x$

Substitute u , du, v and dv into the **Integration by Parts formula** to obtain the result.

Note 2: we reduced our task to integrating $\cos x$ – that is, to integrating **a part of** (the old term for *a factor in*) the original integrand. But note that $\int x \cos xdx \neq \int xdx \cdot \int \cos xdx$ (that is, **an integral of a product is not a product of integrals!**)

Note 3: The only challenge when using integration by parts is deciding which part of the integrand should be denoted by u. The rest is automatic.

2. $\int x^3 \ln x \, dx = \dfrac{x^4 \ln x}{4} - \int \dfrac{x^4}{4}\dfrac{dx}{x} = \dfrac{x^4 \ln x}{4} - \dfrac{x^4}{16} + c$

u - line: $u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}, \ du = \dfrac{dx}{x}$

v – line: $dv = x^3 \, dx \big/ \int, \ v = \dfrac{x^4}{4}$

Note: $\ln x$ is a good choice for u, because u should be differentiated, derivative of $\ln x$ is $\dfrac{1}{x}$ which leads to simplification when multiplied by the integral of power x^3 (which in itself is a power of x).

3. $\int \ln x \, dx = x \ln x - \int x \dfrac{dx}{x} = x \ln x - x + c$

u - line: $u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}, \ du = \dfrac{dx}{x}$

v – line: $dv = dx \big/ \int, \ v = x$

4. $\int x^2 \sin xdx = -x^2 \cos x + \int x \cos xdx =$

u - line: $u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}, \ du = \dfrac{dx}{x}$

v – line: $dv = dx \big/ \int, \ v = x$

Note: x^2 is a good choice for u, because u should be differentiated and derivative of x^2 is $2x$, a lower power, leading to a simpler integrand. Let us continue by evaluating $\int x \cos x dx$ by parts:

u - line: $u = x, \ \frac{du}{dx} = 1, \ du = dx$

v – line: $dv = \cos x \, dx \, / \int, \ v = \int \cos x \, dx = \sin x$

Substitute u , du, v and dv into the **Integration by Parts formula** to obtain

$$= -x^2 \cos x + x \sin x - \int \sin x \, dx = -x^2 \cos x + x \sin x + \cos x + c = \cos x(1 - x^2) + x \sin x + c$$

5. $$\int \underbrace{x^2}_{u} e^{3x} \, dx = x^2 \frac{e^{3x}}{3} - \frac{2}{3} \int e^{3x} \, x dx =$$

$$= x^2 \frac{e^{3x}}{3} - \frac{2}{3} x \frac{e^{3x}}{3} + \frac{2}{9} \int e^{3x} dx = x^2 \frac{e^{3x}}{3} - \frac{2}{9} \frac{e^{3x}}{3} x + \frac{2}{27} e^{3x} + c = \frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$$

6. $$\int \underbrace{e^{3x}}_{u} \sin x \, dx = e^{3x}(-\cos x) + 3 \int \underbrace{\cos x}_{u} \, e^{3x} \, dx =$$

$$e^{3x} \cos x + 3(\sin x e^{3x} - 3 \int e^{3x} \sin x \, dx)$$

$$\Rightarrow \underbrace{\int e^{3x} \sin x dx}_{I} = e^{3x}(3 \sin x - \cos x) - 9 \underbrace{\int e^{3x} \sin x dx}_{I}$$

$$\Rightarrow I = e^{3x} \ (3 \sin x - \cos x) - 9I$$

$$\Rightarrow I = \frac{1}{10} e^{3x} \ (3 \sin x - \cos x) \ + c$$

20.3 Partial fractions

The method of partial fraction is an algebraic trick that allows us to re-write a rational function as a sum of the simplest possible rational fractions. It is an opposite (inverse) operation to adding up fractions using the common denominator method.

Consider $$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \overset{\cancel{\ }(x + a)}{\frac{A}{x - a}} + \overset{\cancel{\ }(x - a)}{\frac{B}{x + a}}$$

$$\Rightarrow \quad A\,(x + a) + B\,(x - a) = 1$$

To solve this equation for A and B use one of the two methods:

I. Remove the brackets

$$A\,x + Aa + B\,x - Ba = 1$$

Collect the like terms

$(A + B)x + Aa - Ba = 1$ for all x

Equate the like coefficients

$$\Rightarrow \begin{cases} A + B = 0 \\ Aa - Ba = 1 \end{cases}$$

$$\Rightarrow \quad A = -B \;\Rightarrow\; -Ba - Ba = 1 \Rightarrow B = -\frac{1}{2a},\; A = \frac{1}{2a}$$

OR

II. Use the killer instinct (!)

$x = -a$: $B\,(-2a) = 1 \Rightarrow B = -\dfrac{1}{2a}$ (chose x to "kill" the first term)

$x = a$: $2aA = 1 \quad \Rightarrow A = \dfrac{1}{2a}$ (chose x to "kill" the second term)

Whatever method you use, you can see that

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

Representing a rational function as a sum of partial fractions is useful e.g. when integrating rational functions.

20.4 Integration of rational functions

Exampe: $\displaystyle\int \frac{1}{(x-2)(x-3)}dx =$

$$\frac{1}{(x-2)(x-3)} = \overset{(x\,-3)}{\frac{A}{x-2}} + \overset{(x\,-\,2)}{\frac{B}{x-3}} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$\Rightarrow \;(A + B)\,x - (3A + 2B) = 1$

Equating the like coefficients,

$A + B = 0 \qquad \Rightarrow \qquad A = -B$

$3A + 2B = -1 \;\Rightarrow\; -3B + 2B = -1 \Rightarrow -B = -1 \Rightarrow B = 1 \Rightarrow A = -1$

$$= \int -\frac{dx}{x-2} + \int \frac{dx}{x-3} = -\ln|x-2| + \ln|x-3| + c = \ln\left|\frac{x-3}{x-2}\right| + c$$

20.5 Decision Tree for Integration

We can now introduce the full Decision Tree for Finding Indefinite Integrals – see figure 20.1:

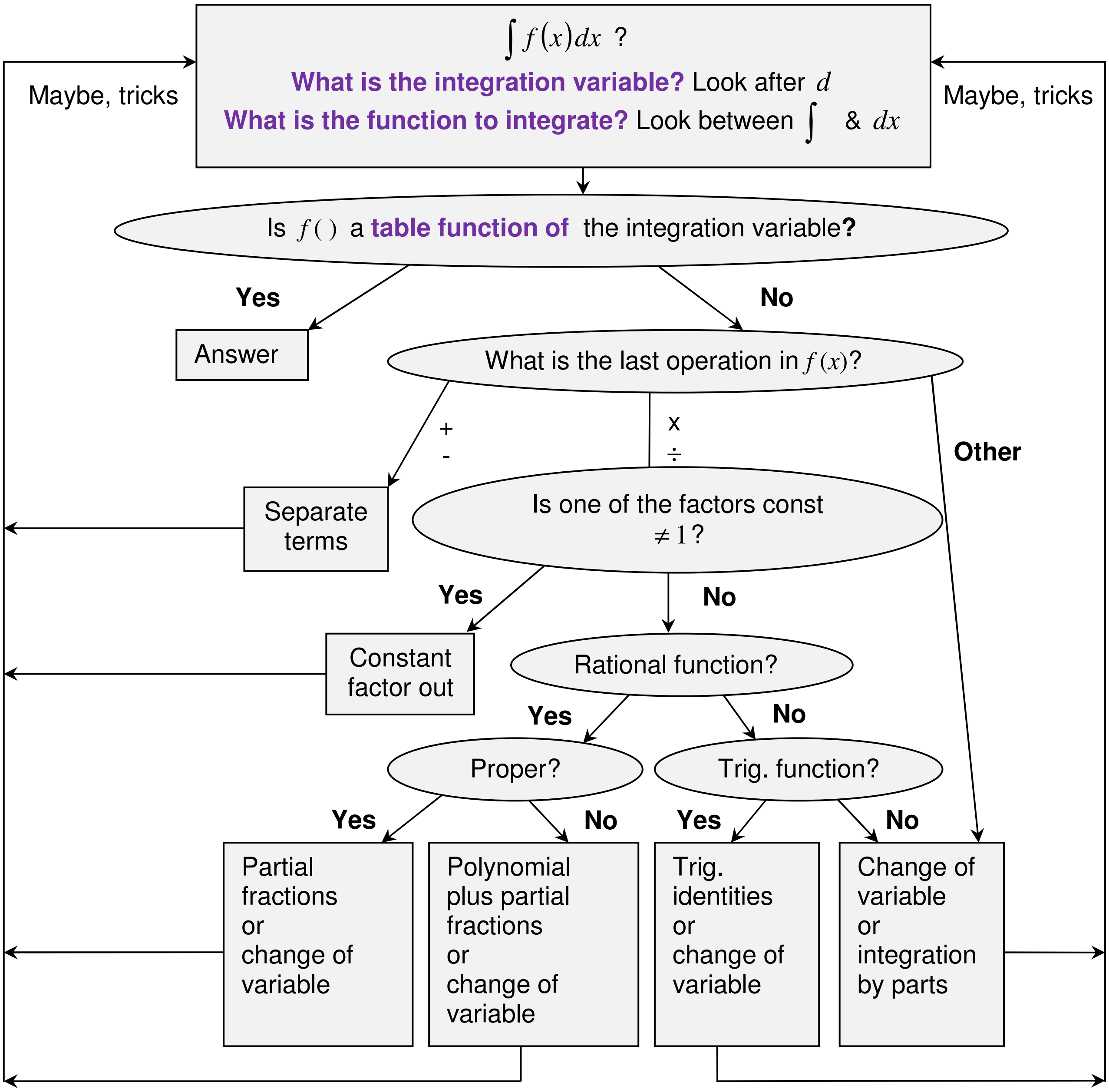


Figure 20.1. Decision Tree for Integration.

20.6 Instructions for self-study

- **Revise ALGEBRA Summary (addition of fractions)**
- **Revise Summaries on FUNCTIONS and TRIGONOMETRY**

- **Revise Lectures 14 - 15 (limits and differentiation)**
- **Revise Lectures 18 - 19 (integration) and study Solutions to Exercises in Lecture 18 using the STUDY SKILLS Appendix**
- **Study Lecture 20 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Find

a) $\int_0^{2\pi} \sin^2 t dt$

b) $\int_0^{2\pi} \cos^2 t dt$

c) $\int \sin x \cos x dx$

d) $\int \cos x \cos 2x dx$

Q2. Find

a) $\int x \sin x dx$

b) $\int x^2 \ln x dx$

c) $\int x^2 e^{3x} dx$

d) $\int e^{2x} \cos x dx$

Q3. Find

a) $\int \frac{1}{(x-1)(x-3)} dx$

b) $\int \frac{x+1}{x^2 - 3x + 2} dx$

c) $\int \frac{x-1}{x^2 - 2x - 8} dx$

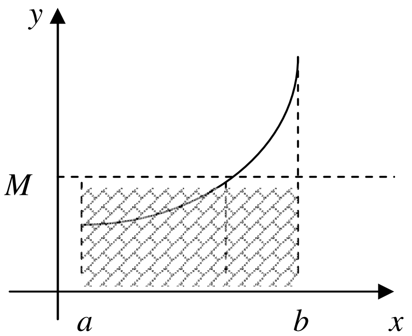
d) $\int \frac{x-1}{x^2 - 2x + 8} dx$

Lecture 21. INTEGRAL CALCULUS: Applications of Integration

21.1 Mean value of a function

Using **The Mean Value Theorem** (Lecture 18), the mean value of a function on the interval between a and b is

$$M \equiv f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{(mean height)}$$



$\Rightarrow$ Area of dashed rectangle = Area under the curve

$$M(b-a) = \int_a^b f(x)dx$$

Example: Find mean of $y = 3x^2 + 4x + 1,\ -1 \le x \le 2$

Solution

$a = -1,\ b = 2$

$$M = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int_{-1}^{2}(3x^2 + 4x + 1)dx = \frac{1}{3}(3x^3 + 2x + x) = 6$$

Optional

RMS value of a function (the square root of the mean value of the squares of y)

Example: Find RMS of $y = x^2 + 3,\ 1 \le x \le 3$

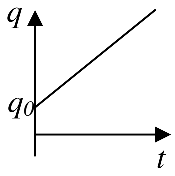
$$= \left[\frac{1}{3-1}\int_1^3 y^2\,dx\right]^{1/2} = \left[\frac{1}{2}\int_1^3 (x^4 + 6x^2 + 9)\,dx\right]^{1/2} =$$

$$= \left[\frac{1}{2}\left(\frac{x^5}{5} + \frac{x^3}{3} + 9x\right)_1^3\right]^{1/2} = \frac{1}{2^{1/2}}\left[\left(\frac{3^5}{5} + \frac{3^3}{3} + 27\right) - \left(\frac{1^5}{5} + \frac{1^3}{3} + 9\right)\right]^{1/2}$$

$$= \frac{1}{2^{1/2}}[74]^{1/2} \approx \sqrt{37} \approx 6$$

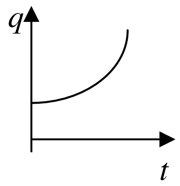
21.2 Electrical systems

If charge q varies linearly with time t, then current i is constant with time,



$i = \frac{q - q_0}{t}$.

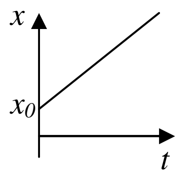
If charge q is non-linear in time t, then instantaneous current i is



$$i = \frac{dq}{dt} \Rightarrow q(t) = \int_0^t i(\tau)d\tau + \underbrace{q(0)}_{q_0}$$

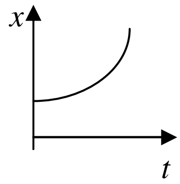
21.3 Mechanical systems

If displacement x varies linearly with time t, then speed v is constant with time,



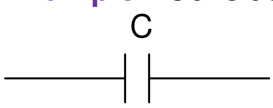
$v = \frac{x - x_0}{t}$.

If displacement x is non-linear in time t, then instant-aneous speed v is



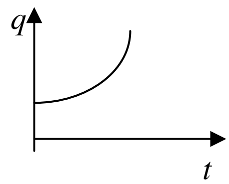
$$\mathrm{v} = \frac{dx}{dt} \Rightarrow x(t) = \int_0^t v(\tau(\tau) \ + \underbrace{x(0)}_{x_0}$$

Example: Consider a capacitor of capacitance C (in *Faradays*).



The voltage drop on the capacitor is

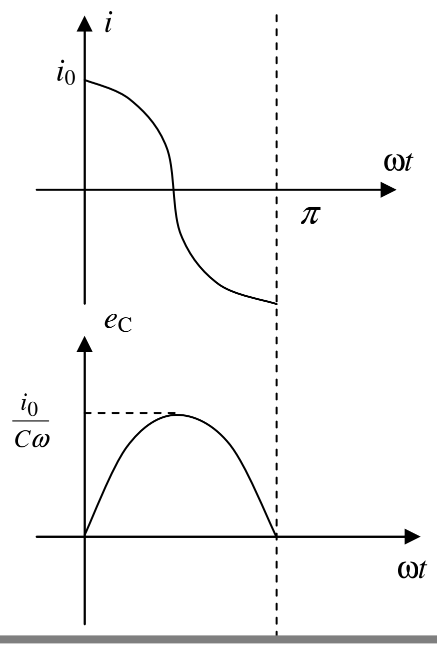
$e_c(\mathrm{t}) = \frac{q(\mathrm{t})}{C} = \frac{1}{C}\int_0^t i(\tau)d\tau + \frac{q_0}{C}$. Below we often assume $q_0 = 0$.



Let $i = i_0 \cos \omega t$, $\omega = \frac{2\pi}{T}$ - circular frequency, T - period

$$\Rightarrow e_c = \int_0^t \frac{i_0}{C} \cos \omega\tau \, d\tau = \frac{i_0}{C\omega} \sin \omega t = \frac{i_0}{C\omega} \cos(\frac{\pi}{2} - \omega t) = \frac{i_0}{C\omega} \cos(\omega t - \frac{\pi}{2})$$

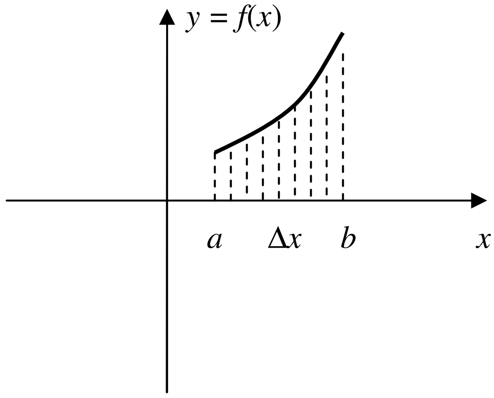
$\Rightarrow$ a capacitor effects a $\frac{\pi}{2}$ shift between current and voltage

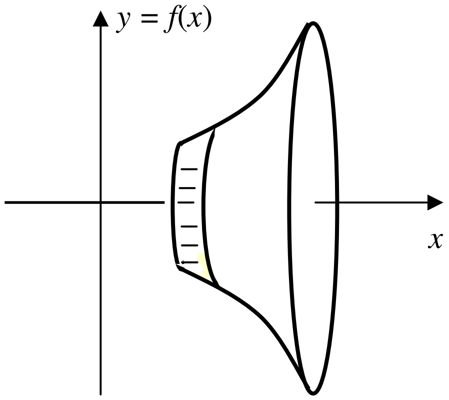


21.4 Rotational systems

21.4.1 Volumes of Solids of Revolution

If the plane figure bounded by the curve $y = f(x)$, x-axis and straight lines $x = a$ and $x = b$ rotates through a complete revolution about the x-axis, it generates a solid. Let V be its volume.





To find V we represent the area under the curve as a sum of areas of thin strips of width Δx (see figure above). The volume is approximately a sum of volumes of thin cylinders generated by each of these thin strips rotating about the x-axis.,

Question: What is the volume of a cylinder?
Answer:

Question: In our case, what is the radius r at the base of each thin cylinder?
Answer:

Question: What is its height h of each thin cylinder?
Answer:

$\Rightarrow V \approx \sum_n \Delta V = \sum_n \pi y^2 \Delta x \rightarrow \pi \int_a^b y^2 dx$ as $\Delta x \rightarrow 0$.

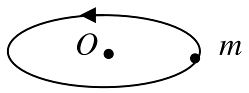
Example: Find the volume generated when the plane figure bounded by $y = 5\cos 2x$, x-axis and straight lines $x = 0$ and $x = \dfrac{\pi}{4}$ rotates through a complete revolution about the x-axis.

$$V = \pi \int_a^b y^2 dx = \pi \int_0^{\pi/4} [5\cos(2x)]^2 dx = \pi \int_0^{\pi/4} 25\cos^2(2x) dx = 25\pi \int_0^{\pi/4} \frac{1+\cos(4x)}{2} dx =$$

$$12.5\pi \left[x + \frac{\sin(4x)}{4} \right] \Bigg|_0^{\frac{\pi}{4}} = 12.5\pi \left[\frac{\pi}{4} + \frac{\sin(4 \cdot \frac{\pi}{4})}{4} \right] \approx 3.1\pi^2 \approx 30$$

21.4.2 The moment of inertia

The moment of inertia I of a point of mass m rotating around a center O is $I = mr^2$, where r is the distance between the point and the centre.

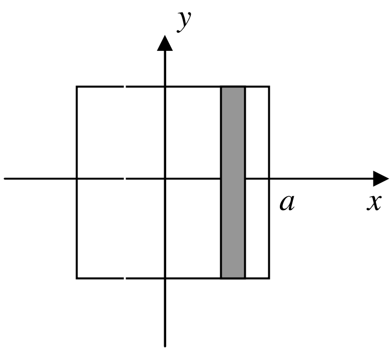


The moment of inertia of a lamina around an axis is approximately the sum of moments of inertia $\Delta I = r^2\Delta m$ of elementary points of mass Δm lying at the distance r from the axis of rotation,

$$I \approx \sum_n \Delta I = \sum_n r_n^2 \Delta m \to \int_{surface} r^2 dm \text{ as } \Delta x \to 0.$$

Example:

Let a square lamina rotate around the y-axis (see figure below). All points in this lamina are at the same distance r from the axis.



If **mass density** (mass per unit area), ρ, of the lamina is constant (does not change with position), then the mass of the shaded strip is $\Delta m = 2\rho a \Delta x$ (density times area), and the moment of inertia of the lamina is

$$I = \int_{surface} r^2 dm = \int_{-a}^{a} x^2 2\rho a dx = 2\rho a \left. \frac{x^3}{3} \right|_{-a}^{a} = \frac{2\rho a}{3}(a^3 - (-a)^3) = \frac{4\rho a^4}{3} = \frac{Ma^2}{3}$$

where the lamina mass is $M = 4\rho a^2$.

Question: Why is the lamina mass $4\rho a^2$?
Answer:

21.5 Instructions for self-study

- **Revise Lectures 14 - 15 (differentiation)**
- **Revise Lectures 19 and study Solutions to Exercises in Lecture 19 using the STUDY SKILLS Appendix**
- **Study Lecture 20 using the STUDY SKILLS Appendix**
- **Study Lecture 21 using the STUDY SKILLS Appendix**

- **Do the following exercises:**

Q1. Given $i(t) = i_0 \sin \omega t$ find e_C and e_L - provided $q_0 = -\frac{i_0}{\omega}$, that is, $e_C(0) = \frac{q_0}{C} = -\frac{i_0}{\omega C}$.

Q2. Find the volume generated when the plane figure bounded by $y = 4x$, x-axis and straight lines $x = 0$ and $x = 1$.

Q3.

a) The moment of inertia about a diameter of a sphere of radius 1 *m* and mass 1 *kg* is found by calculating the integral

$$\frac{3}{8}\int_{-1}^{1}(1-x^2)^2\,dx.$$

Show that the moment of inertia of the sphere is $\frac{2}{5}$ *kg m*^{2}.

b) Calculate the moment of inertia of a uniform thin rod of mass M and length l about a perpendicular axis of rotation at its end.

Lecture 22. Ordinary Differential Equations

22.1 Basic concepts

Consider the equation

$$\frac{dy(x)}{dx} = ay(x) \tag{22.1}$$

Eq. (22.1) is an example of an **ordinary differential equation**. It is called an **equation**, because it is a mathematical statement that contains the = sign and that can be true or false depending on which **function** you substitute for the unknown $y(x)$ (compare to the case of an algebraic equation $P_n(x) = 0$: an algebraic equation is a mathematical statement that contains the = sign and can be true or false depending on which **value** you substitute for the unknown x).

Terminology: $y(x)$ is called an unknown (function) or dependent variable**;** x – an independent variable; a is a (constant) coefficient.

Eq. (22.1) is a **differential** equation, because it contains a derivative of the unknown y. It is called **ordinary,** because the unknown is a function of one variable (the differential equations with unknowns being functions of several variables are called **partial**, because they contain partial derivatives).

To solve a differential equation means to find all **functions** which turn it into a true statement (compare to the case of an algebraic equation: to solve an algebraic equation means to find all **values** of the unknown variable which turn it into a true statement).

Examples:

1. In the following equations, state which variable is dependent and which variable is independent:

a) $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t$. **Answer:**

b) $RC\frac{dv}{dt} + v = v_0$ **Answer:**

c) $y\frac{dy}{dt} + c(t)y = 0$. **Answer:**

2. Solve $\frac{dy(x)}{dx} = e^x$

The above equation is simple – only one term contains the unknown function $y(x)$. Therefore, it can be solved using the Decision Tree for Solving Simple Equations.
In general, to solve a differential equation it is important to classify it first, because different types of differential equations can be solved using different methods (tricks).
The differential equations are classified according to their order, whether they are linear or non-linear and whether their coefficients are constant or variable.

22.2 Order of a differential equation

The **order of a differential equation** is the order of its highest derivative.

Example:
Establish the order of the above differential equations a) – c). **Answers:**

22.3 Linearity or non-linearity of a differential equation

A differential equation is called **linear** if each of its terms either contains no unknown function or is a product of a known coefficient (constant or variable) and the unknown function or one of its derivatives.

Example:
Establish which of the above differential equations a) – c) is linear and which non-linear
Answers:

22.4 Differential equations with constant coefficients

A differential equation has **constant coefficients if** all coefficients (co-factors) of the dependent variable and its derivatives are constant (with respect to the independent variable).

Example:
Establish which of the above differential equations a) – c) has constant coefficients.
Answer:

22.5 Homogeneous and inhomogeneous ODEs

A differential equation is called **homogeneous** if each of its terms contains the unknown or one of its derivatives. Otherwise it is called **inhomogeneous**.

Example
Establish which of the differential equations a) – c) is homogeneous. **Answer:**

22.6 The first order linear homogeneous equation with constant coefficients

Eq. (22.1) is a general form of the first order linear homogeneous equation with constant coefficients. We can check by substitution that $y = e^{ax}$ is its solution and so is $y = c\, e^{ax.}$ with any arbitrary constant factor c.

$\Rightarrow$ Any **first** order linear homogeneous equation with constant coefficients has infinitely many solutions involving **one** arbitrary constant.

22.7 The initial value problems

In reality, the initial value of an unknown function is often known. If given an initial condition, say $y(0) = y_0$, the solution becomes unique.

The problem

$$\begin{cases} y' = ay \\ y(0) = y_0 \end{cases}$$

is an example of an **Initial Value Problem**.

22.8 Balance equations in chemical engineering

For every element of a chemical system there will be a material balance equation for each chemical species present:

$$\frac{d(\text{quantity})}{dt} = \sum(\text{inflow rates}) - \sum(\text{outflow rates})$$

In this equation *quantity* must be an extensive quantity, e.g. mass (*kg),* energy*(Joules) etc.* Do not try to write material balance equations for temperature or concentration. Flow rates must be measured, respectively, in *kg/s, Watts*, etc.

Example:

Consider a tank containing V *litres* of a solution consisting of x_0 *(kg*) of salt dissolved In water. At the initial moment of time $t = 0$, let us start pumping pure water into the tank at the rate of r (*litres/s*) and let $x(t)$ *(kg*) be the mass of salt in the tank at any moment t. Let us keep the mixture uniform by stirring. Let the outflow rate of mixture be the same as the inflow rate of water, that is, r. Then the volume V of mixture is kept constant throughout. It is easy to check that under these conditions the outflow rate of salt is

$$r\frac{x(t)}{V} \ (kg/s)$$

and the amount of salt $x(t)$ satisfies the ordinary differential equation

$$x' = -r\frac{x(t)}{V}$$

The associated Initial Value Problem is

$$\begin{cases} x' = -\dfrac{r}{V}x(t) \\ x(0) = x_0 \end{cases}$$

22.9 Ordinary differential equations with complex coefficients

Consider an equation

$$y' = jy \tag{22.2}$$

Using Section 22.6, the function $y = y_0\, e^{j\,t}$ is its solution.

Optional

We can also prove this without using Section 22.6 as follows:

Substituting $y = y_1 + jy_2$ into the left-hand side of Eq. (22.2) we obtain:

$$y' = y_1' + jy_2'$$

Substituting $y = y_1 + jy_2$ into the right-hand side of Eq. (22.2) we obtain:

$$jy = j\,y_1 - y_2$$

This means that Eq. (22.2) can be written as

$$y_1' + jy_2' = jy_1 - y_2 \tag{22.3}$$

At each moment t, Eq. (22.3) has one complex number on the left-hand side and another, in the right. This means that their real parts are equal and so are their imaginary parts,

$$y_1' = -y_2$$

$$y_2' = y_1$$

Let $y_2 = y_0 \sin t \Rightarrow y_1 = y_2' = y_0 \cos t$. This means that

$$y = y_1 + jy_2 = y_0 \cos t + j\, y_0 \sin t = y_0\, e^{jt},$$

where we used the Euler's formula.

Note: When differentiating or integrating, complex coefficients must be treated in the same manner as real.

22.10 Applications

Eq. (22.1) describes many systems in which at any given time, the rate of change of the variable (that is, derivative of the variable) is proportional to the value of this variable.

Ordinary differential equations and associated Initial Value Problems are common in engineering, since engineering systems affect the transformation of signals which often involve their rate of change, that is their derivatives. In engineering units most of the time you use linear ordinary differential equations of second order, with constant coefficients.

22.11 Instructions for self-study

- **Revise Decision Tree for Solving Simple Equations**
- **Revise Summaries on COMPLEX NUMBERS and DIFFERENTIATION**
- **Revise Lecture 20 and study Solutions to Exercises in Lecture 20 using the STUDY SKILLS Appendix**
- **Revise Lecture 21 using the STUDY SKILLS Appendix**
- **Study Lecture 22 using the STUDY SKILLS Appendix**
- **Do the following exercises:**

Q1. Verify that

a) $y = 3\sin 2x$ is a solution of $y'' + 4y = 0$,

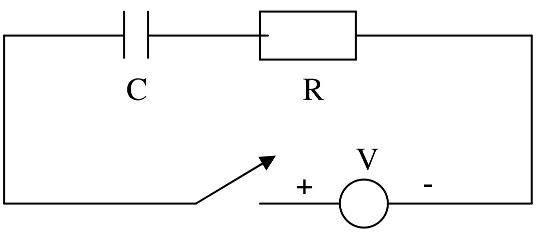
b) $2e^x$ satisfies equation $y'' + 4y = 0$,

c) Ae^x satisfies equation $y'' - 2y' + y = 0$,

d) $Ae^x + Bxe^x$ satisfies equation $y'' - 2y' + y = 0$.

e) Classify all the above equations.

Q2. Consider a circuit, comprising a capacitor of capacitance C and a resistor R placed in series:



When a constant voltage source, V, is applied it can be shown that the current, I, through the circuit satisfies the ordinary differential equation.

$$R\frac{dI}{dt} + \frac{1}{C}I = 0$$

Given the initial current values $I(0) = VR$ solve the corresponding initial value problem, that is, find the current $I(t)$ through this circuit.

Q3. Consider a tank containing 5 *litres* of a solution consisting of 0.5 (kg) of salt dissolved in water. Let pure water be pumped into the tank at the rate of 1 (*litres*/s) and the mixture, which is kept uniform by stirring, be pumped out at the same rate. Find $x(t)$, the amount of salt in the tank at any moment *t*.

Q4. Solve the equations

a) $y' = 2jy$

b) $y'' = 2\sin x$

IV. SUMMARIES

Algebra Summary

OPERATIONS

Addition (direct operation)

Addition of whole numbers gives whole number

1. $a + b = b + a$

Terminology: a and b are called **terms**
$a + b$ is called **sum**

2. $(a + b) + c = a + (b + c)$

Subtraction (inverse operation)

Def : $a - b = x$**:** $x + b = a$

Note: $a + b - b = a$ **(subtraction undoes addition)**
$a - b + b = a$ **(addition undoes subtraction)**

3. $a + 0 = a$

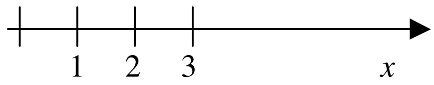
4. for each $\underline{a}$ there exists one **additive inverse** $-a$:
$a + (-a) = 0$

Rules (follow form Laws):

$+(b + c) = +b + c$
$+\ a + b = a + b$
$-(-a) = a$
$-(a) = -a$

TYPES OF VARIABLES

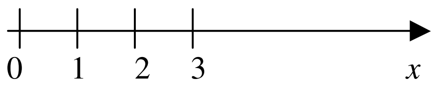
Whole numbers are $1, 2, 3, ...$



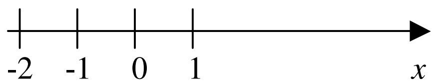
introduces **0** and **negative numbers**:

$a - a = 0$
if $b > a$ $a - b = -(b - a)$

Natural numbers are $0, 1, 2, ...$



Integers are ..., -2, -1, 0, 1, 2, ...



Multiplication (direct operation)

For whole numbers n

$$a\cdot n = \underbrace{a+...............+a}_{n \text{ times}}$$

Notation: $ab = a\cdot b = a\times b$

$2b = 2\cdot b = 2\times b$

$23 \neq 2\cdot 3,\ 23 = 2\cdot 10+3$

$2\frac{1}{2} \neq 2\cdot\frac{1}{2},\ 2\frac{1}{2} = 2+\frac{1}{2}$

$2\frac{3}{2} = 2\cdot\frac{3}{2}$

1. $a\cdot b = a\cdot b$

Terminology: a and b are called **factors**

ab - **product**

2. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

Conventions: $abc = (ab)c$

$a(-bc) = -abc$

3. $a(b+c) = ab+ac$

$\longrightarrow$ Removing brackets

$\longleftarrow$ Factoring

4. $a\cdot 0 = 0$
5. $a\cdot 1 = a$

Rules (follow from Laws):

$(a + b)(c + d) = ac + ad + bc + bd$ (**SMILE RULE**)

$(-1)\cdot n = -n$

$(-1)\cdot(-1) = 1$

Division (inverse operation)

Def: $a/b = x$: $xb = a$

Terminology: a – **numerator**

b – **denominator**

a/b – **fraction (ratio)**

proper fraction if $|a|<|b|$, a, b integers

Note: $ab/b = a$ **(division undoes multiplication)**

$(a/b)b = a$ **(multiplication undoes division)**

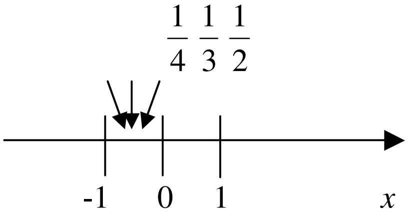
6. For each $a \neq 0$ there exists one **multiplicative inverse** $1/a$: $a\cdot 1/a = 1$

introduces **rational numbers**

Def: Rationals are all numbers $\frac{m}{n}$,

where m and $n \neq 0$ are integers

(**division by zero is not defined**)



Rules:

$$\frac{a}{b}\cdot n = \frac{an}{b}$$

$$\frac{a/b}{n} = \frac{a}{bn} = \frac{a/n}{b}$$

$$\frac{an}{bn} = \frac{a\cdot \cancel{n}}{b\cdot \cancel{n}} = \frac{a}{b}$$ **CANCELLATION**

$$\frac{\frac{1}{n}}{m} = \frac{m}{n}$$ **FLIP RULE**

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

Note: $\frac{a+c}{b} = (a+c)/b$

$$\frac{\overset{d}{a}}{b} + \frac{\overset{b}{c}}{d} = \frac{ad}{bd} + \frac{cb}{db} = \frac{ad+cb}{bd}$$

COMMON DENOMINATOR RULE

n-th **power** b^n (direct operation)
If n – a whole number
$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}}$$

Rules

$a^m \cdot a^n = a^{m+n}$
(product of powers with the same base is a power with indices added)
$a^n \cdot b^n = (ab)^n$
(product of powers is power of product)
$a^m/a^n = a^{m-n}$
(ratio of powers is power of ratios)
$a^m/b^m = (a/b)^m$
(ratio of powers with the same base – subtract indices)
$a^0 = 1$
$a^{-n} = 1/a^n$
$(a^m)^n = a^{mn}$ [Convention: $a^{m^n} = a^{(m^n)}$]

n-th **root** (inverse to taking to power n)

Def: $\sqrt[n]{b} = x$: $x^n = b$
Note:
$\sqrt[n]{b^n} = b$
(taking *n-th* root undoes taking *n-th* power)
$(\sqrt[n]{b})^n = b$
(taking *n-th* power undoes taking *n-th* root)

Therefore, can use **notation** $b^{1/n} = \sqrt[n]{b}$

(Indeed, $\sqrt[n]{b^n} = (b^n)^{1/n} = b^{n \cdot \frac{1}{n}} = b^1 = b$)

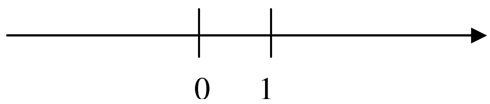
Logarithm base b (inverse to taking b to power)

Def: $\log_b a = n$: $b^n = a$

Note:
$\log_b b^n = n$ (check using definition: $b^n = b^n$)
(taking $\log_b$ undoes taking b to power)
$b^{\log_b n} = n$
(taking b to power undoes $\log_b$)

introduces **irrational** (not rational) **numbers** $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, …

Real numbers are all rationals and all irrationals combined. Corresponding points cover the whole real line



introduces **irrational** (not rational) **numbers**, $\log_{10} 2$, $\log_{10} 3$, *etc.*

Roots and logs also introduce **complex** (not real) **numbers,** $\sqrt{-1}, \log_{10}(-1)$, *etc.*

Rules (follow from Rules for Indeces):

$\log_b xy = \log_b x + \log_b y$

(log of a product is sum of logs)

$\log_b x/y = \log_b x - \log_b y$

(log of a ratio is difference of logs)

$\log_b 1 = 0$ **(log of** 1 **is** 0**)**

$\log_b b = 1$ **(**$b^1 = b$**)**

$\log_b 1/a = -\log_b a$

$\log_b x^n = n \log_b x$

(log of a power is power times log)

$\log_b a = \log_c a / \log_c b$

(changing base)

General remarks

1. $a - b = a + (-b)$ $\longrightarrow$ a difference can be re-written as a sum

2. $\frac{a}{b} = a \cdot \frac{1}{b} = ab^{-1}$ $\longrightarrow$ a ratio can be re-written as a product

3. $\sqrt[n]{b} = b^{1/n}$ $\longrightarrow$ a root can be re-written as a power

4. All laws and rules of addition, multiplication and taking to integer power operations apply to real numbers.

5. Operations of addition, subtraction, multiplication, division (by non-zero) and taking to integer power when applied to real numbers produce real numbers. Other algebraic operations applied to real numbers do not necessarily produce real numbers.

6. By convention, in any algebraic expression operations should be performed using the Order of Operations convention.

Functions Summary

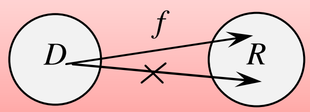
Variables are denoted mostly by $x, y, z, p, \ldots\ldots w$.
A variable can take any value from a set of allowed numbers.

Functions are denoted mostly by f, g and h or $f(\,)$, $g(\,)$ and $h(\,)$
(no multiplication sign is intended!).

In mathematics, the word **function** has two meanings:

1) $f()$ - an operation or a chain of operations on an **independent variable (argument)*;***
2) $f(x)$ – a **dependent variable** (a variable dependent on x)**,** that is the result of applying the operation $f(\,)$ to an independent variable x.

A diagrammatical representation of a function



To specify a function we need to specify a (series of) operation(s) and **domain** D (an allowed set of values of the independent variable). To each $x \in D$, $f(x)$ assigns one and only one value $y \in R$ (**range**, the set of all possible values of the dependent variable).

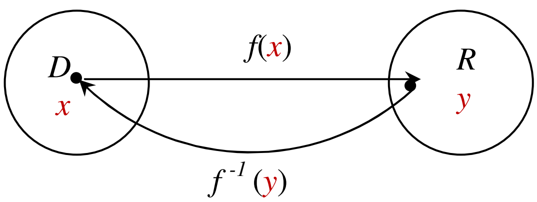
Inverse functions

$f^{-1}(x)$**:** $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ (the function and inverse function undo each other)

symbol of inverse function, not a reciprocal

The inverse function does not always exist!

A Diagrammatic Representation of Inverse Function



Order of Operations Summary

When **evaluating** a mathematical **expression** it is important to know the order in which the **operations** must be performed. The **Order of Operations** is as follows:

First, expression in **Brackets** must be **evaluated**. If there are several sets of brackets, e.g. {[(   )]}, expressions inside the inner brackets must be **evaluated** first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. **Everything raised and everything lowered is considered as bracketed**, and some authors do not bracket **arguments** of elementary **functions**, such as $\exp$, $\log$, $\sin$, $\cos$, $\tan$ *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.*

Other **operations** must be performed in the order of decreasing complexity, which is
oiB - operations in Brackets (including implicit)
F - Functions $f(\ )$
P - Powers (including inverse operations of roots and logs)
M - Multiplication (including inverse operation of division)
A - Addition (including inverse operation of subtraction)

That is, the more complicated **operations** take precedence. For simplicity, we refer to this convention by the abbreviation **oiBFPMA**.

Order of operations (OOO)

1. Make implicit (invisible) brackets visible (everything raised and everything lowered is considered to be bracketed and so are function arguments)
2. Perform operations in brackets {[()]} first (inside out)

OOO

oiB	f( )	P	M	A
(including implicit)		$(\ )^{(\ )}$	$\times$	+
		roots	$\div$	-
		logs		

Quadratics Summary

A **quadratic expression** is a general polynomial of degree 2 traditionally written as

$$ax^2 + bx + c,$$

where a is the constant factor in the quadratic term (that is, the term containing the independent variable squared): b is a constant factor in the linear term (that is, the term containing the independent variable) and c is the free term (that is, the term containing no independent variable).

A **quadratic equation** is the polynomial equation

$$ax^2 + bx + c = 0.$$

Its two **roots** (solutions) can be found using the standard formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Once the roots are found the quadratic expression can be **factorised** as follows:

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

Trigonometry Summary

Conversion between degrees and radians

An angle described by a segment with a fixed end after a full rotation is said to be $360°$ or 2π (radians)

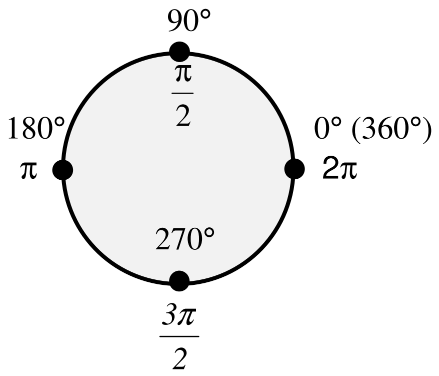
$$\Rightarrow 2\pi\ (rad) = 360°$$
$$\Rightarrow 1\ (rad) \approx 57°$$

The radian is a dimensionless unit of angle.

$$\Rightarrow\ x\,(rad) = x\,(rad)\,\frac{180^{o}}{\pi(\text{rad})} = y^{o}\ ,\quad y^{o} = y^{o}\,\frac{\pi(\text{rad})}{180^{o}} = x\,(rad)$$

Usually, if the angle is given in radians the units are not mentioned (since the radian is a dimensionless unit).

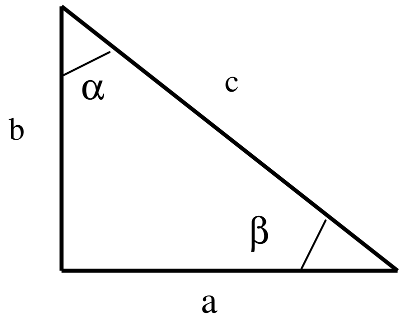
Frequently used angles



$30° = (\frac{30\pi}{180} =)\ \frac{\pi}{6}$ $60° = (\frac{60\pi}{180} =)\ \frac{\pi}{3}$ $120° = (\frac{120\pi}{180} =)\ \frac{2\pi}{3}$	$45° = \frac{\pi}{4}$

Right Angle Triangles and Trigonometric Ratios

Trigonometric ratios sin, cos and tan are defined for **acute angles** (that is, angles less than 90^0) as follows:



$$\sin\alpha = \cos\beta = \frac{a}{c}$$

$$\cos\alpha = \sin\beta = \frac{b}{c}$$

$$\tan a = \cot\beta = \frac{a}{b}$$

$\alpha + \beta = 90^0$ and α and β are called **complementary angles**

Frequently used trigonometric ratios

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$	$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Trigonometric identities

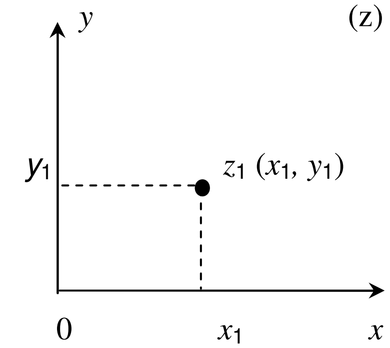
$\sin^2 x + \cos^2 x = 1$
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2\sin x \cos x$$

$A\cos x + B\sin x = \frac{1}{\sqrt{A^2 + B^2}}\sin(x+\alpha)$, where $\tan\alpha = \frac{A}{B}$

Complex Numbers

The Cartesian representation of a complex number

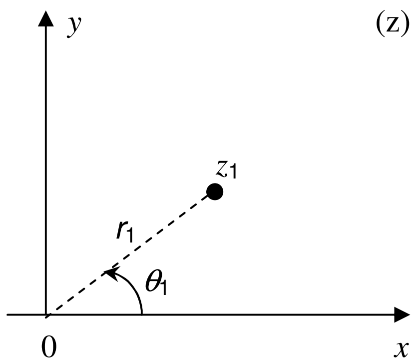


real part, Re z **imaginary part**, Im z (on the Argand diagram)

$z = x + jy$, where x, y – real, $j = \sqrt{-1}$

x-coordinate y-coordinate (on the ordinary plane)

The exponential representation of a complex number



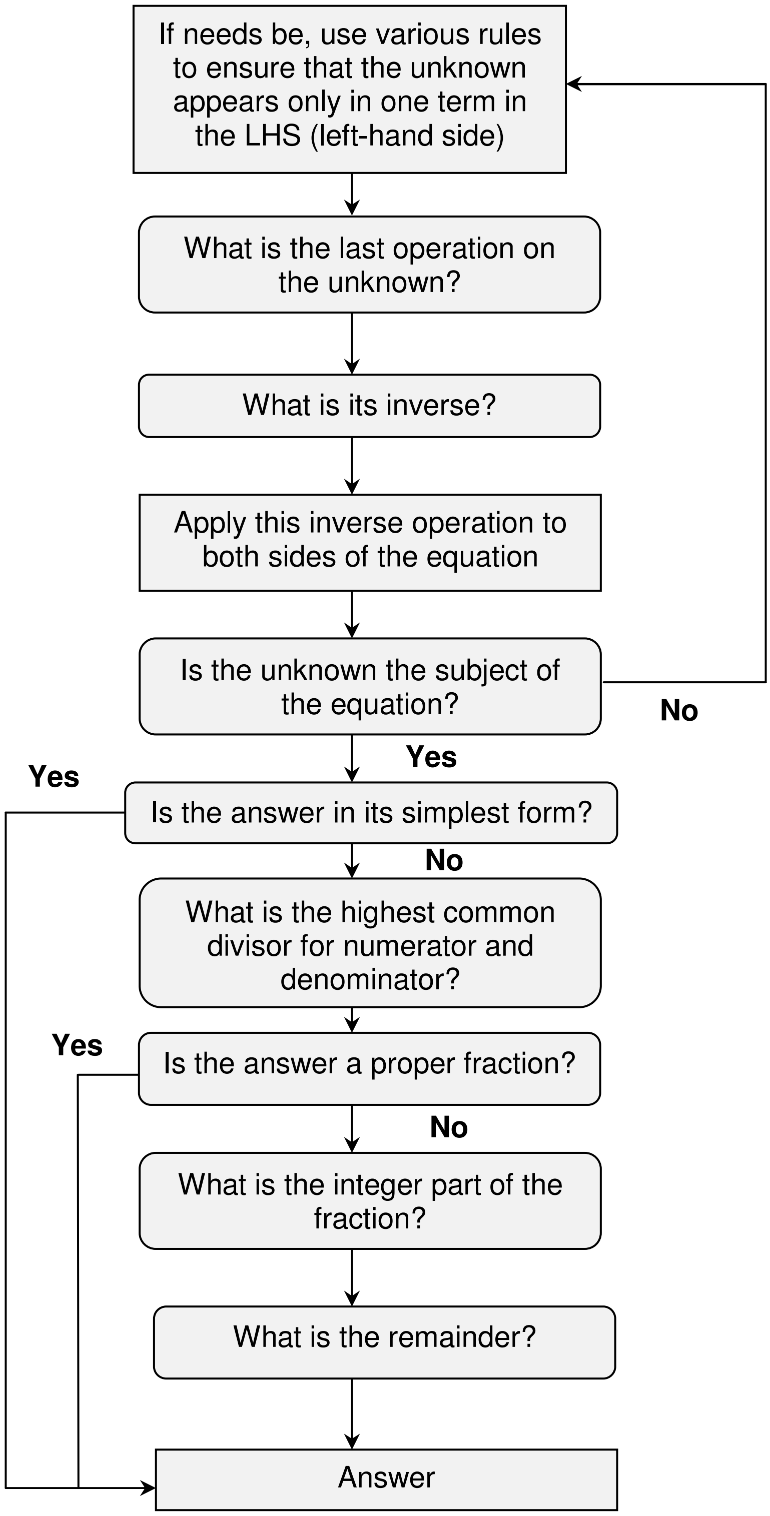
magnitude, $|z|$ **argument,** arg(z) (on the Argand diagram)

$z = r\, e^{j\theta}$, where $r \geq 0$, θ - angle
(in degrees or radians)

polar radius r
(distance to the origin)

polar angle θ (on the ordinary plane)

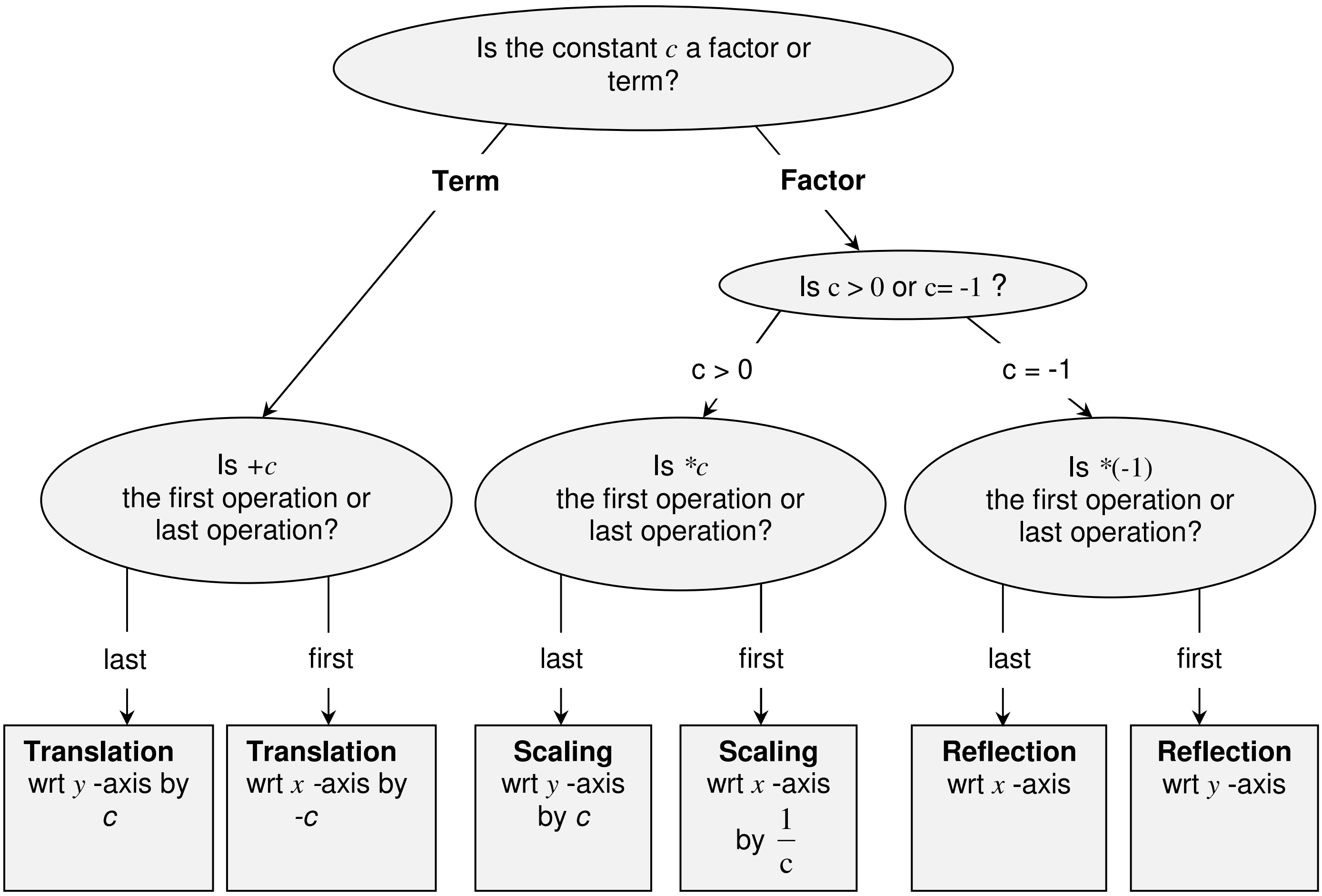
Decision Tree For Solving Simple Equations



<u>Sketching Graphs by Simple Transformations</u>

1. Drop all constant factors and terms.
2. Bring the constants back one by one in the Order of Operations (not necessary but advisable) and at each Step use the Decision Tree given below to decide which simple transformation is affected by each constant. Sketch the resulting graphs underneath one another.

DECISION TREE FOR SKETCHING $y=f(x)+c$, $y=f(x+c)$, $y=cf(x)$and $y=f(cx)$



Note 1: if c is a **negative factor**, write $c = (-1)*|c|$, so that c affects two simple transformations and not one.
Note 2: if c is affects **neither first operation nor last**, the Decision Tree is not applicable. Try algebraic manipulations.
Note 3: if **the same operation** is applied to x in all positions in the equation this operation still should be treated as the first operation, as in $\ln(x - 2)+\sin(x - 2)$.
Note 4: if y is given **implicitly rather than explicitly**, so that the equation looks like $f(x,y) = 0$, then in order to see what transformation is effected by adding a constant c to y or multiplying it by a constant c, y should be made the subject of the functional equation. Therefore, similarly to transformations of x above, the corresponding transformation of y is defined by the inverse operation, $-c$ or $\frac{1}{c}$, respectively.

Finding a Limit of a Sequence

To find a limit of a sequence (as $k \to \infty$) use the first principles (graphical representation) or

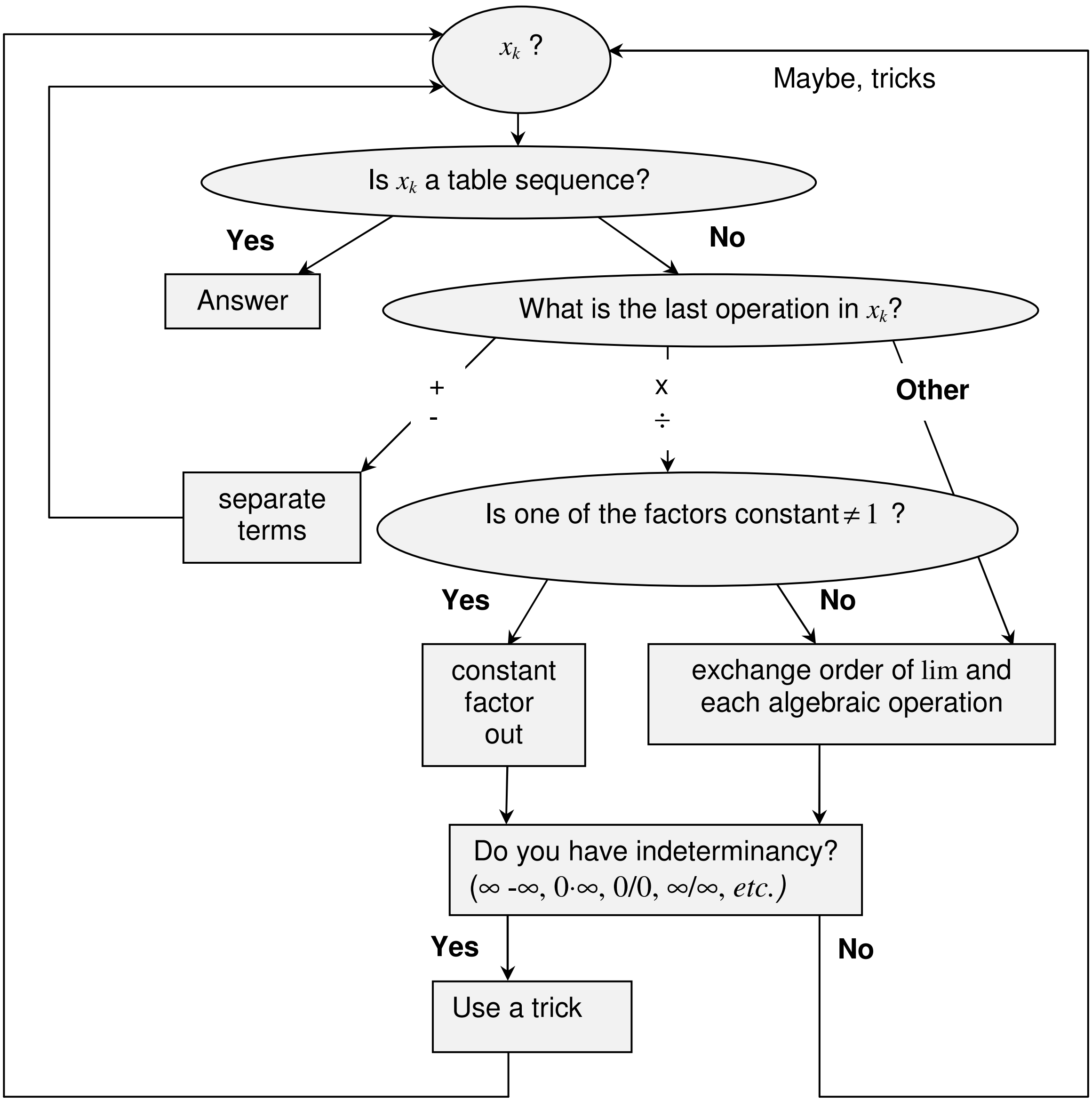
TABLE

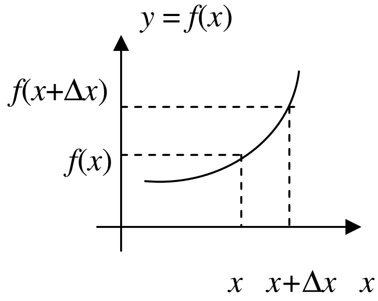
x_k	$\lim x_k$
constant	Constant
k	∞
$\frac{1}{k}$	0
$\frac{1}{a^n}$ $(a > 0)$	1
q^n $(\lvert q \rvert < 1)$	0

RULES

1. $\lim \alpha x_k = \alpha \lim x_k$ – *const factor out*
2. $\lim [x_k \pm y_k] = \lim x_k \pm \lim y_k$ – *sum rule (separate terms)*
3. $\lim (x_k\, y_k) = \lim x_k \lim y_k$ – *product rule*
4. $\lim \frac{x_k}{y_k} = \frac{\lim x_k}{\lim y_k}$ – *quotient rule*
5. $\lim f(x_k) = f(\lim x_k)$
   limit and any algebraic operation or simple function can exchange places (commute)

DECISION TREE





Differentiation Summary

To differentiate a function use the first principles

$$\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

or represent roots and fractions as powers, make invisible brackets visible and use

TABLE

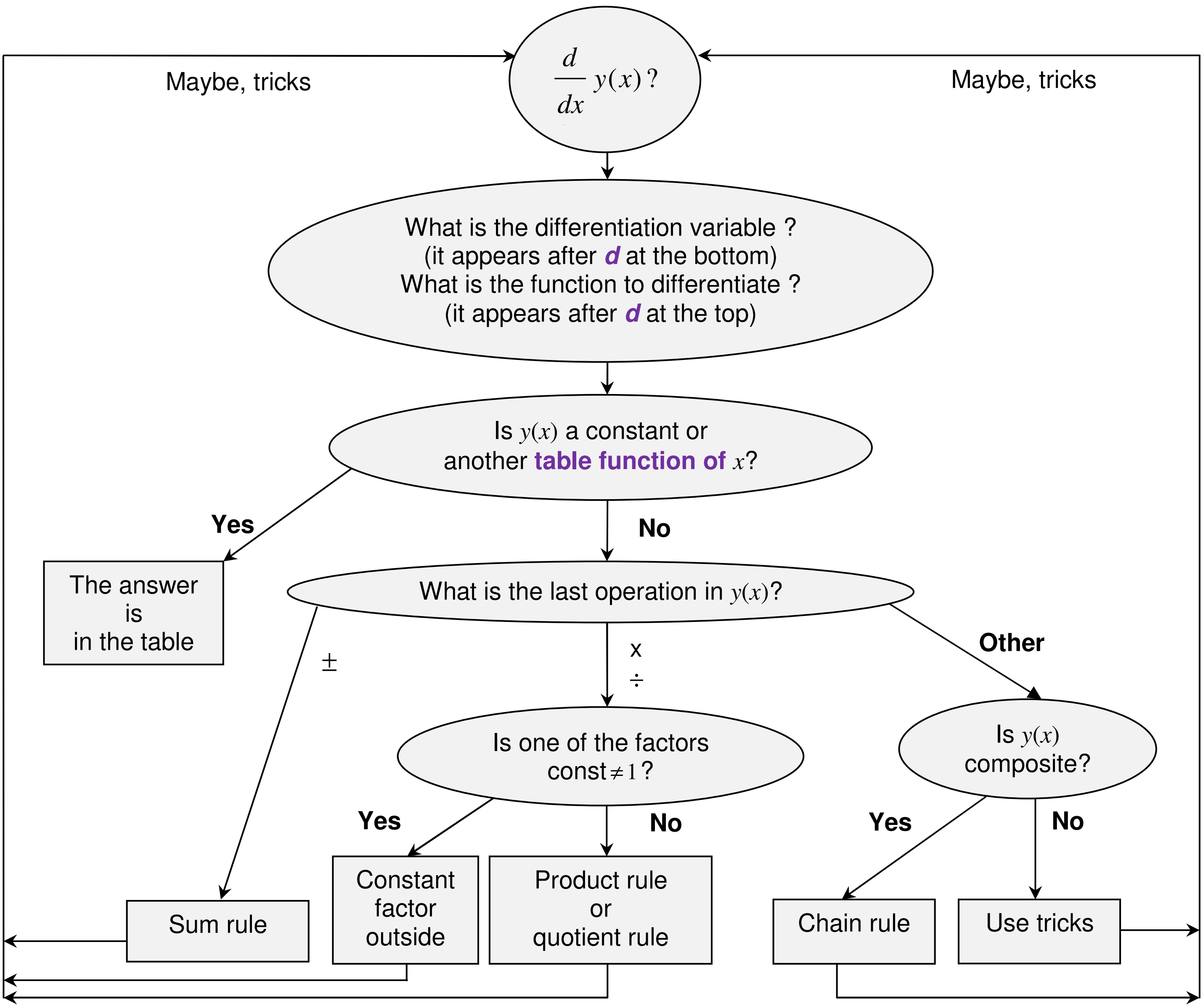
$f(x)$	$\frac{d}{dx}f(x)$
constant	0
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

RULES

1. $\frac{d}{dx}[\alpha f(x)]=\alpha\frac{df(x)}{dx}$ - *const factor out*
2. $\frac{d}{dx}[f(x)+g(x)]=\frac{df(x)}{dx}+\frac{dg(x)}{dx}$ – *sum rule (separate terms)*
3. $\frac{d}{dx}[f(x)g(x)]=\frac{df(x)}{dx}g(x)+\frac{dg(x)}{dx}f(x)$ – *product rule*
4. $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{\frac{df(x)}{dx}g(x)-\frac{dg(x)}{dx}f(x)}{g^2(x)}$ – *quotient rule*
5. $\frac{d}{dx}[f(g(x))]=\frac{df(g(x))}{dg(x)}\frac{dg(x)}{dx}$ – *chain rule*

(decompose, differentiate, multiply)

DECISION TREE

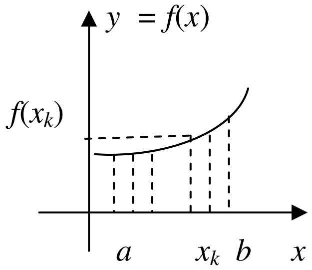


Integration Summary

To find a definite integral of a function use the definition

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{x_k = a}^{b} f(x)\Delta x$$

or represent roots and fractions as powers, make invisible brackets visible and use



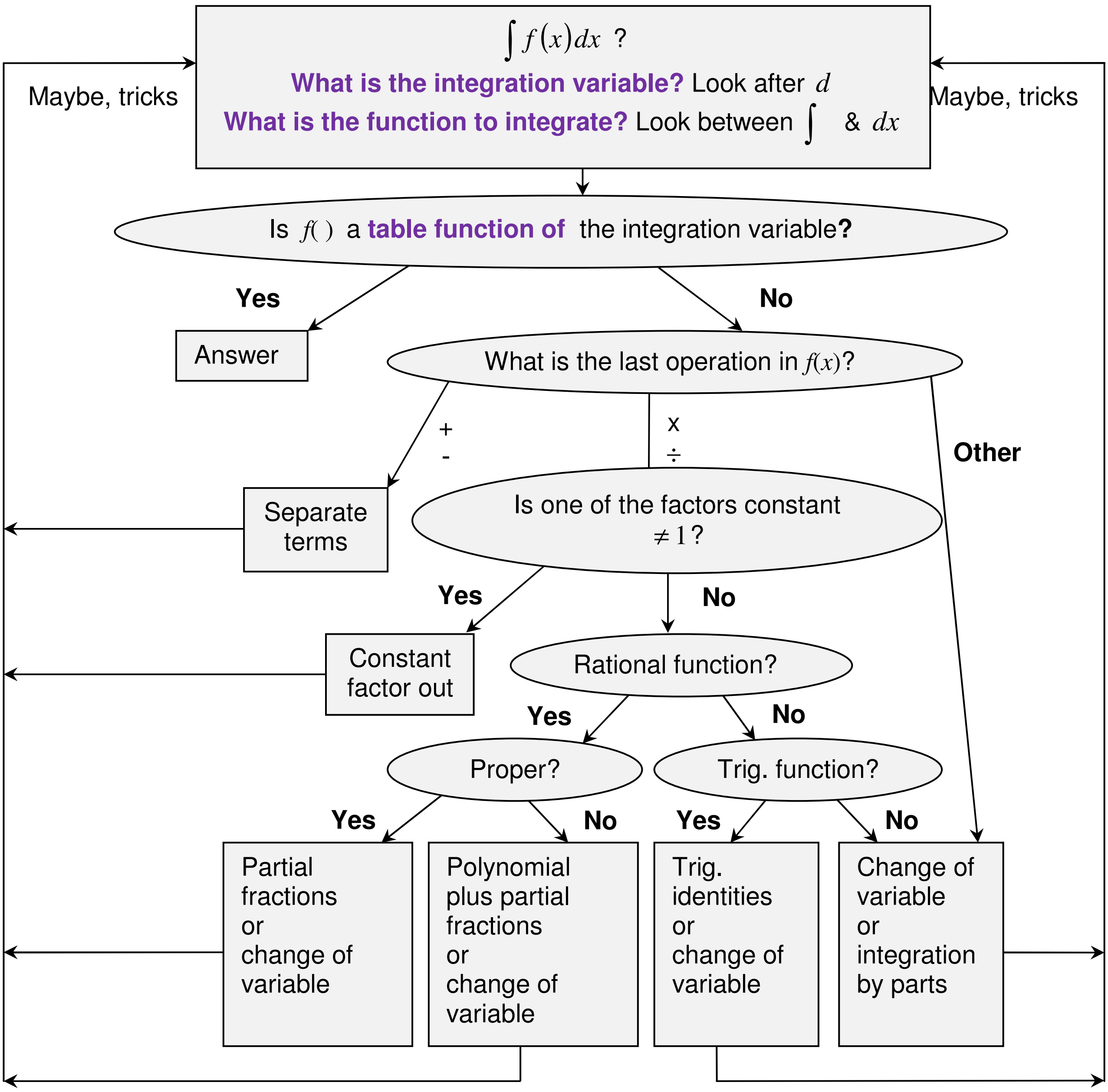
TABLE

$f(x)$	$F(x) = \int f(x)dx$
0	c
e^x	$e^x + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$x^n,\ n \neq 1$	$\dfrac{x^{n+1}}{n+1} + c$
$\dfrac{1}{x}, x \neq 0$	$\ln\lvert x\rvert + c$

RULES

1. $\int \alpha f(x)dx = \alpha \int f(x)dx$ - *constant factor out*
2. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$ - *sum rule (separate terms)*

DECISION TREE



V. GLOSSARY

ABSTRACTION - a general concept formed by extracting common features from specific examples.

ACUTE ANGLE – a positive angle which is smaller than 90^0.

ALGEBRAIC OPERATION – OPERATION of addition, subtraction, multiplication, raising to power, extracting a root (surd) or taking a log.

ALGORITHM – a sequence of solution steps.

ARGUMENT – INDEPENDENT VARIABLE, INPUT.

CANCELLATION (in a numerical fraction) – operation of dividing both numerator and denominator by the highest common DIVISOR.

CIRCLE – a LOCUS of points inside and including a CIRCUMFERENCE.

CIRCUMFERENCE – a LOCUS of points at the same distance from a specific point (called a centre).

COEFFICIENT – a CONSTANT FACTOR multiplying a VARIABLE, e.g. in the EXPRESSION $2ax$, x is normally a VARIABLE and $2a$ is its COEFFICIENT.

COMPOSITION – a combination of two or more functions (operations) forming a single new function by applying one to the output of another, e.g. composition of two function '$f\,(\,)$' and '$g\,(\,)$' can be represented as $f\,(g\,(\,))$.

CONCEPT - a technical word or phrase.

CONSTANT – a number or a mathematical quantity that can take a range of (numerical) values but is independent of the main CONTROL VARIABLE – ARGUMENT or UNKNOWN, e.g. in the EXPRESSION x + a, x is usually understood to represent a VARIABLE and a, a constant (which is independent of this VARIABLE).

If there is only one algebraic CONSTANT, the preferred choice for its algebraic symbol is a. The second choice is b, and the third, c. If there are more CONSTANTS in the EXPRESSION, then one chooses, in the order of preference, d and e, and then upper case letters in the same order of preference. If more CONSTANTS are required, we make use of subscripts and superscripts and Greek letters.

DIAGRAM - a general (abstract) visualisation tool, a pictorial representation of a general set or relationship.

DIFFERENCE – a mathematical expression in which the last operation is subtraction.

DIMENSIONAL QUANTITY – a quantity measured in arbitrary units chosen for their convenience, such as *s, m, N, A, m/s,* kg/m^3, £.

DOMAIN – a set of all allowed values of the ARGUMENT.

EQUATION – a mathematical statement involving VARIABLES and the = sign which can be true or false, depending on the values taken by the VARIABLES (which are called UNKNOWNS in this case), e.g. $2x + 3y = 10$ is true when $x = y = 2$ but not when $x = y = 1$.

EVALUATE – find the (numerical) value of a (mathematical) EXPRESSION.

EXPLICIT – 1) clearly visible; 2) a subject of equation.

EXPONENTIATION – a mathematical operation of raising to power.

EXPRESSION (mathematical) – a combination of numbers, brackets, symbols for variables and symbols for mathematical operations, e.g. $2(a + b)$, $2ab$.

FACTOR – a (mathematical) EXPRESSION which multiplies another (mathematical) EXPRESSION, e.g. *ab* is a PRODUCT of two FACTORS, *a* and *b*.

FINAL (SIMPLEST) FORM (of a numerical fraction) – no CANCELLATIONS are possible, and only PROPER FRACTIONS are involved.

FORMULA – a mathematical statement involving VARIABLES and the = sign which and is always true.

FREE TERM – a CONSTANT TERM.

FUNCTION – a mathematical OPERATION or a composition of OPERATIONS which establishes a relationship between values of the ARGUMENTS (INDEPENDENT VARIABLES, INPUTS) in its DOMAIN and VALUE OF THE FUNCTION (DEPENDENT variable, output). a function is completely defined only when its domain is defined, e.g. $f(x) = 2x + 3$, x - real' describes a FUNCTION whose SYMBOL is 'f' or '$f(\ \ )$' (no multiplication is implied) and whose ARGUMENT is called 'x'. Its DOMAIN is 'all reals'. Given any value of 'x', you can find the corresponding value of this FUNCTION by first multiplying the given value of 'x' by 2 and then by adding 3 to the result.

FUNCTION SYMBOL – usually a Latin letter, usually from the first part of the alphabet. If there is only one FUNCTION, the preferred choice for its symbol is 'f'. The second choice is 'g', and the third, 'h'. If there are more variables, then one chooses, in order of preference, 'u', 'v' and 'w'. If more FUNCTIONS are involved, we often make use of subscripts and superscripts, upper case and Greek letters.

GENERALISATION - an act of introducing a general concept or rule by extracting common features from specific examples.

GRAPH - a specific visualisation tool, a pictorial representation of a particular set or relationship.

IDENTITY – the same as FORMULA.

IMPLICIT – not EXPLICIT.

INDEPENDENT VARIABLE – ARGUMENT, INPUT, e.g. given '$y = f(x)$', 'x' is an INDEPENDENT VARIABLE.

INPUT – (value of) ARGUMENT, (value of) INDEPENDENT VARIABLE, e.g. given '$y = f(x)$', 'x' is INPUT; also given '$y = f(2)$', '2' is INPUT.

INTEGER PART – when dividing a positive integer m into a positive integer n, k is the INTEGER PART if it is the largest positive integer producing $k*m \le n$. The REMAINDER is the difference $n - k*m$, e.g. when dividing 9 into 2, the INTEGER PART is 4 and the REMAINDER is 1, so that $\frac{9}{2} = 4 + \frac{1}{2} = 4\frac{1}{2}$.

INVERSE (to an) **OPERATION** – operation that (if it exists) undoes what the original OPERATION does

LAST OPERATION – see ORDER OF .OPERATIONS.

LHS – Left Hand Side of the EQUATION or FORMULA, to the left of the '='-sign

LINEAR EQUATION – an EQUATION which involves only FREE TERMS and TERMS which contain the UNKNOWN only as a FACTOR, e.g. '$2x - 3 = 0$' is a LINEAR EQUATION, '2' is a COEFFICIENT in front of the UNKNOWN and '-3' is a FREE TERM.

LOCUS – a set of all points on a plane (or in space) with a specific property.

NECESSARY - 'A' is a NECESSARY condition of 'B' IF 'B $\Rightarrow$ A' (B implies A), so that 'B' cannot take place unless 'A' is satisfied.

NON-DIMENSIONAL QUANTITY - a quantity taking any value from an allowed set of numbers.

NON-LINEAR EQUATION – an EQUATION which is not LINEAR, e.g. '$2 \ln(x) - 3 = 0$' is a NON-LINEAR EQUATION.

OPERATION (mathematical) – something that can be done to CONSTANTS and VARIABLES. When all CONSTANTS and VARIABLES entering an EXPRESSION are given values, OPERATIONS are used to EVALUATE this EXPRESSION.

ORDER OF OPERATIONS When EVALUATING a (mathematical) EXPRESSION it is important to know the order in which the OPERATIONS must be performed. By convention, the ORDER OF OPERATIONS is as follows: First, expression in BRACKETS must be EVALUATED. If there are several sets of brackets, e.g. {[(   )]}, expressions inside the inner brackets must be EVALUATED first. The rule applies not only to brackets explicitly present, but also to brackets, which are implied. Two special cases to watch for are fractions and functions. Indeed, when $(a + b)/(c + d)$ is presented as a two-storey fraction the brackets are absent, and some authors do not bracket ARGUMENTS of elementary FUNCTIONS, such as exp, log, sin, cos, tan *etc.* In other words, e^x should be understood as $\exp(x)$, $\sin x$ as $\sin(x)$ *etc.* Other OPERATIONS must be performed in the order of decreasing complexity, which is

FUNCTIONS $f(\ )$

POWERS (including inverse operations of roots and logs)

MULTIPLICATION (including inverse operation of division)

ADDITION (including inverse operation of subtraction)

That is, the more complicated OPERATIONS take precedence.

OUTPUT – (value of) DEPENDENT VARIABLE, (value of the) FUNCITON, e.g. given '$y = f(x)$', 'y' is OUTPUT; also given '$f(x) = 2x + 3$' and '$x = 2$', '7' is OUTPUT (indeed, $2*2+3 = 7$).

PRODUCT – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is multiplication, x, e.g. 'ab' is a PRODUCT, and so is '$(a + b)c$'.

QUOTIENT - a mathematical expression where the last operation is division.

RANGE – the set of all possible values of the function. **HERE**

REARRANGE EQUATION, FORMULA, IDENTITY – the same as TRANSPOSE.

REMAINDER – see INTEGER PART.

RHS – Right Hand Side of the EQUATION or FORMULA, to the right of the '=' sign.

ROOT OF THE EQUATION – SOLUTION of the EQUATION.

SEQUENCE – a function with an INTEGER ARGUMENT.

SIMPLE EQUATION – an EQUATION OF ONE UNKNOWN, LINEAR or NON-LINEAR but such that can be reduced to LINEAR by a simple CHANGE OF VARIABLE.

SIMPLE TRANSFORMATIONS - translation, scaling or reflection - are affected by adding a constant or multiplying by a constant.

SOLUTION OF AN ALGEBRAIC EQUATION – constant values of the UNKNOWN VARIABLE which turn the EQUATION into a true statement.

SOLVE – find SOLUTION of the EQUATION.

SUBJECT OF THE EQUATION – the unknown is the SUBJECT OF THE EQUATION if it stands alone on one side of the EQUATION, usually, LHS.

SUBSTITUTE – put in place of.

SUFFICIENT - 'A' is a SUFFICIENT condition of 'B' IF 'A $\Rightarrow$ B' (A implies B), so that if 'A' is satisfied, then 'B' takes place.

SUM – a (mathematical) EXPRESSION in which the LAST operation (see the ORDER OF OPERATIONS) is addition, +, e.g. $a + b$ is a SUM, and so is $a(b + c) + ed$.

TERM – a (mathematical) EXPRESSION that is added to another (mathematical) EXPRESSION, e.g. $a + b$ is a SUM of two TERMS, a and b.

TRANSPOSE EQUATION, FORMULA, IDENTITY – make a particular unknown the subject of EQUATION, FORMULA, IDENTITY, so that it stands on its own in the LHS or RHS of the corresponding mathematical statement.

UNKNOWN – a VARIABLE whose value can be found by solving an EQUATION, e.g. in equation $x + 2 = 3$, x is an UNKNOWN.

VALUE – a number.

VARIABLE – a mathematical quantity that can take a range of (numerical) values and is represented by a mathematical symbol, usually a Latin letter, usually from the second part of the alphabet. If there is only one VARIABLE, the preferred choice for its algebraic symbol is x. The second choice is y and the third, z. If there are more variables, then, one chooses, in the order of preference, letters u, v, w, s, t, r, p and q, then the upper case letters in the same order of preference. If more VARIABLES are required, we make use of subscripts, superscripts and Greek letters.

VI. STUDY SKILLS FOR MATHS

Assuming that you have **an average background** in mathematics you need to study these notes on your own for **6 hours** each week:

1. **Spend half an hour revising the Summary or Summaries suggested for Self Study.** You should be able to use Order of Operations, algebraic operations and Decision Trees very fast. Do not forget to keep consulting the **Glossary**.

2. **Spend 2.5 hours revising previous Lectures and Solutions to Exercises.**

3. **Spend 1.5 hours studying the latest Lecture** (see tips below on how to do that).

4. **Spend 1.5 hours doing the exercises given in that lecture for self-study.**

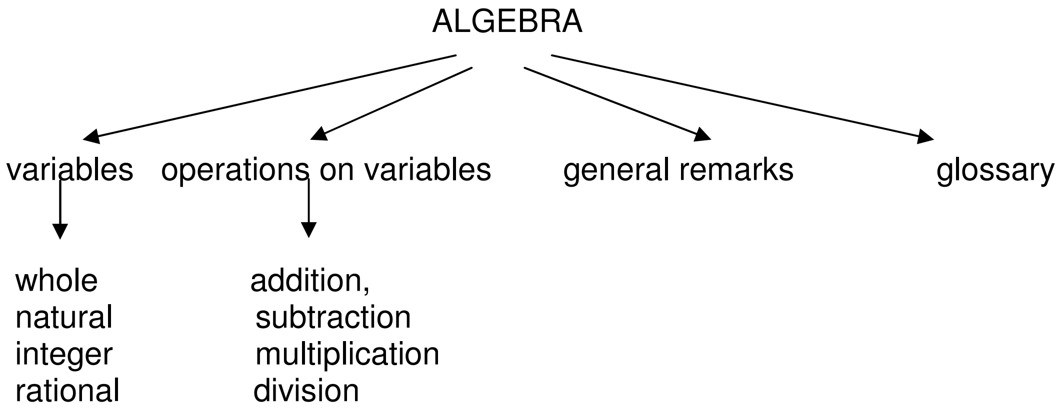
Some of you need to study 12 hours a week. Then multiply each of the above figures by two!

This is how to study each new lecture:

1. Write down the topic studied and list all the subtopics covered in the lecture. Create a flow chart of the lecture. This can be easily done by watching the numbers of the subtopics covered.

For example in Lecture 1 the main topic is ALGEBRA but subtopics on the next level of abstraction are 1.1, 1.2, …, 1.3.

Thus, you can construct the flow chart which looks like



2. Write out the glossary for this lecture; all the new words you are to study are presented in bold red letters.

3. Read the notes on the first subtopic **several times** trying to understand how each problem is solved

4. Copy the first problem, put the notes aside and try to reproduce the solution. **Be sure in your mind that you understand what steps you are doing.** Try this a few times before checking with the notes which step is a problem.

5. Repeat the process for each problem.

6. Repeat the process for each subtopic.

7. Do exercises suggested for self-study

Here are a few **tips on how to revise for a mathematics test or exam:**

1. Please study the **Summaries** first.

2. Keep consulting the **Glossary**.

3. Then study **Lectures and Solutions** to relevant Exercises one by one in a manner suggested above.

4. Then go over **Summaries** again.

Here is your **check list**: you should be thoroughly familiar with

1. The **Order of Operations** Summary and how to “make invisible brackets visible”.

2. The **Algebra Summary**, in particular, how to **remove brackets, factorise and add fractions**. You should know that **division by zero is not defined**. Rules for logs are secondary. You should know what are integers and real numbers. You should not "invent rules" with wrong cancellations in fractions. You should not invent rules on changing order of operations, such as addition and power or function. You should all know the precise meaning of the words **factor, term, sum and product**.

3. The concept of **inverse operation**.

4. The **Decision Tree for Solving Simple Equations**.

5. The **formula for the roots of the quadratic equation** and how to use them **to factorise any quadratic**.

6. The diagrammatic representation of the function (see the **Functions Summary**). You should know that a function is an operation (or a chain of operations) plus domain. You should know the meaning of the words argument and domain. You should know what is meant by a real funciton of real variable (real argument). You should know the precise meaning of the word constant (you should always say "constant with respect to (the independent variable) x or t or whatever..."

7. How to do function composition and decomposition using **Order of Operations**

8. How to use graphs

9. How to sketch elementary functions: the straight line, parabola, exponent, log, sin and cos

10. The **Trigonometry Summary**

11. The approximate values of e (≈ 2.71) and π (≈ 3.14)

12. What is j $(=\sqrt{-1})$ and what is j^2 $(=-1)$

13. The Cartesian and exponential form of a complex number and how to represent a complex number on the Argand digram (the **Complex Numbers Summary**)

14. How to add, multiply, divide complex numbers, raise them to integer and fractional power

15. The **Differentiation Summary**

16. The **Integration Summary**

17. The **Sketching by Simple Transformation Summary**

18. The **Limits Decision Tree** and basic indeterminacies

19. Sketching by analysis

20. The definition of a **mean of a function on an interval**

VII. TEACHING METHODOLOGY (FAQs)

Here I reproduce a somewhat edited correspondence with one of my students who had a score of about 50 in his Phase Test and 85 in his exam. You might find it instructive.

Dear Student

The difference between stumbling blocks and stepping stones is how you use them!

Your letter is most welcome and helpful. It is extremely important for students to understand the rationale behind every teacher's decision. All your questions aim at the very heart of what constitutes good teaching approach. For this reason I will answer every one of your points in turn in the form of a Question - Answer session:

Q: I have understood the gist of most lectures so far. However there have been a number of lectures that towards the end have been more complicated and more complex methods were introduced. When faced with the homework on these lectures I have really struggled. The only way I have survived have been to look at the lecture notes, Croft's book, Stroud's book and also various websites.

A: Any new topic has to be taught this way: simple basic facts are put across first and then you build on them. If you understand simple facts then the more sophisticated methods that use them seem easy. If they do not this means that you have not reached understanding of basics. While in general, reading books is extremely important, at this stage I would advise you to look at other books only briefly and only as a last resort, spending most of the time going over the lectures over and over again. The problem with the books available at this level is that they do not provide too many explanations. LEARNING IS A CHALLENGING AND UNINTUITIVE POCESS. IF YOU BELIEVE THAT YOU UNDERSTAND SMETHING IT DOES NOT MEAN THAT YOU DO!

Q: Many of the homework questions are way beyond the complexity of any examples given in lectures. Some are or seem beyond the examples given in books.

A: None of them are, although some could be solved only by very confident students who are already functioning on the level of the 1st class degree. There are four important points to be aware of here:

1. If you have not reached the 1st class level yet, it does not mean that you cannot reach it in future.

2. 1st class degree is desirable to be accepted for a PhD at elite Universities, others as well as employers are quite happy with 2.1.

3. It is absolutely necessary for students to stretch themselves when they study and attempt more challenging problems than they would at exams, partly because then exams look easy.

4. Even if you cannot do an exercise yourself, you can learn a lot by just trying and then reading a solution.

Q: This has been and continues to be demoralising.

A: A proper educational process is a painful one (no pain no gain!), but it also should be enlightening. One of the things you should learn is how to "talk to yourself" in order to reassure yourself. One of the things that I have been taught as a student and find continually helpful is the following thought: "Always look for contradictions. If you find a contradiction (that is, see that there is something fundamentally flawed in your understanding) - rejoice! Once the contradiction is resolved you jump one level up in your mastery of the subject (problem)." In other words, you should never be upset about not understanding something and teach yourself to see joy in reaching new heights.

Q: Take this week as an example. Exercises suggested in Lecture 1 look to me like 'integration by parts ' which I learned in Croft's book. However in your notes these seem to be 'separate terms'. There were not any examples given in the lecture on ' integration by parts' or by ' separate terms ' and I am not sure if we were meant to be using the 'substitution method', if this is even possible.

A: This example illustrates very well why at this stage reading books brings more harm than good. There are no questions in Lecture 19 which require "integration by parts", only the techniques that have been discussed in that Lecture, that is, "separate terms" rule, "constant factor out" rule and "substitution" method. It is true that there were no examples in the lectures on how to "separate terms". This omission is due to the fact that sometimes when during a lecture, I omit very simple steps. They are included into the written lectures.
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Q: I also feel that the pace has been too high; we spent only about an hour on sequences and 2 weeks on integration.

A: We have only 22 weeks in one year to cover the necessary material, and it is spread out accordingly. However, you do not **need** to know more about sequences than was already discussed.

Q: When revising each week it is most unnatural to have to take your mindset back a week or two to try to remember what you have learned at a certain stage. I really do not think that many do it.

A: This question touches on one of the most fundamental aims of education - development of long term memory. Both short-term memory and long-term memory are required to be a successful student and a successful professional. When I ask you to memorise something (and say that this is best done by going over the set piece just before going to bed) I am exercising your short-term memory. How can you develop a long-term memory, so that what we study to-day stays with you - in its essence - for ever? The only way to do that is by establishing the appropriate connections between neural paths in your brain. If you have to memorise just a sequence of names, facts or dates there are well established techniques promoted in various books on memory. They suggest that you imagine a Christmas tree or a drive-in to your house, imagine various objects on this tree or along the drive-way and associate the names, facts or dates with this objects. However, this technique will not work with technical information. What you need to establish are much deeper - meaningful - connections. The only way to do this is to go over the same material again and again, always looking at it from a new vantage point. While your first intuitive reaction is that "it is most unnatural to have to take your mindset back a week to try to remember what you have learned at a certain stage", this is the only proper way to learn a technical subject and develop your long term memory.

Q: Related to this is the strange system where only the specified method can be used to derive an answer. At our level I feel that any method which produces the correct answer should be accepted. If you have been used to doing something one way and are forced to change then, for students that are a bit weak anyway, this will be a problem.

A: This is actually a classical educational technique, aiming at two things at once:

1. practicing certain methods and techniques,

2. developing students' ability to "work to specs".

People who do not come to terms with this idea are going to have problems with the exam questions where the desired techniques are specified. They will lose most of the marks if they use another technique.

Q: The principles based on the first principles I feel are out of the question at our level in S1 and should be left to S2 as a minimum.

A: These proofs are above the A-level, but are definitely the 1st year level. Everyone should be able to start these proofs on the right note, namely write out the appropriate definitions and then SUBSTITUTE the appropriate functions. Only the students who reached the 1st class level are expected to finish these proofs using various tricks, but everyone should be able to follow the proofs as given by their teachers in Solutions to Exercises and learn how the tricks work. Also, remember, that in real life you will hardly ever differentiate or integrate using differentiation or integration techniques, but remembering the definitions (a derivative is a local slope or a local rate of change and integral is a signed area between a curve and a horizontal axis) might come very handy in your engineering life.

Q: Techniques should be introduced into the whole system to help build confidence, although I realise that there is a balance to be struck.

A: Techniques are introduced according to the internal logic of the material. But confidence building is important and this is something teachers and students have to work at together. Teachers unfortunately have little time for that, all we can do is keep saying "good, good" when progress is made. You spend more time with yourself, so please keep reminding yourself that Exercises are only there to help to learn. What is important is that you are constantly stretching yourself. Please keep reminding yourself how much you achieved already. Surely, there are lots of things you can do now that you could not even dream of doing before. An extremely important educational point that you are touching upon here is the following: the so-called liberal system of education that was introduced in the 60s (and consequences of which we all suffer now) provided only "instant gratification". What the real education should be aiming at is "delayed gratification". You will see the benefits of what you are learning now - in their full glory - LATER, in year 2 and 3, not to-day.

Hope this helps!